On the Conjugacy of Element-Conjugate Homomorphisms II

Michael Larsen*
University of Pennsylvania
Philadelphia, PA 19104

§0. Introduction

This paper continues the effort begun in [3] to classify the acceptable compact Lie groups G. We recall that two homomorphisms $\phi_1, \phi_2 : \Gamma \to G$ are *element-conjugate* when $\phi_1(\gamma)$ is conjugate in G to $\phi_2(\gamma)$ for all $\gamma \in \Gamma$. We say two homomorphisms are *globally conjugate* if there exists an element $g \in G$ such that

$$\phi_2(\gamma) = g\phi_1(\gamma)g^{-1}$$

for all $\gamma \in \Gamma$. A Lie group G is *acceptable* if element-conjugate homomorphisms from a finite group Γ to G are globally conjugate. The classification problem seems hopelessly difficult if we allow disconnected groups G. Already many finite groups are unacceptable; consider, for example, the following homomorphisms $(\mathbb{Z}/2\mathbb{Z})^2 \to S_6$:

$$(1,0) \mapsto (12)(34),\ (0,1) \mapsto (13)(24); \quad (1,0) \mapsto (12)(34),\ (0,1) \mapsto (12)(56).$$

We therefore consider only connected groups, which we often choose to regard as connected reductive algebraic groups over $\mathbb{R}$.

This paper classifies connected, simply connected, acceptable, compact groups. As a byproduct of the method, we also treat the non-simply connected, exceptional groups. It turns out that the only acceptable, exceptional group is G_2. The original motivation for this question is the construction of automorphic forms appearing with multiplicity greater than 1 in the space of cusp forms on the dual group of G [1]. Our methods also find application in differential geometry; we construct for each $n \geq 35$, a non-isometric pair of isospectral manifolds with universal covering space $\mathrm{Spin}(n)$.

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§1. Notations and Terminology

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(1.1) Unless otherwise specified, an *algebraic group* means a reductive linear algebraic group over $\mathbb{R}$ with a compact group of $\mathbb{R}$-points. If X is the name of a root system, we abuse notation by letting the same symbol denote the semisimple, connected, simply connected algebraic group associated with this root system. We write X^{ad} for the adjoint group of the same type, *i.e.*, the quotient of the algebraic group X by its center. To avoid confusing unitary groups with alternating groups, we denote the latter $\mathcal{A}_n$. We also use the notations $\mathrm{SU}(n)$, $\mathrm{Spin}(n)$, $\mathrm{SO}(n)$, *etc.* to denote compact real Lie groups. By $\mathrm{Lie}(X)$, we mean the complexified Lie algebra of $X(\mathbb{R})$. The superscript $^\circ$, applied to an algebraic group denotes identity component. A *subgroup* H of G means an algebraic group H endowed with a homomorphism $H \to G$ which is injective at the level of complex points. The *intersection* of subgroups H_1 and H_2 of G, denoted $H_1 \cap H_2$, means the fibre product $H_1 \times_G H_2$. Note that the intersection of two subgroups is again an algebraic group in our sense because its group of real points is again compact.

(1.2) By a *minimal representation* of a compact Lie group, we mean an almost faithful representation of minimal dimension. For simple groups, this turns out to be unique up to outer automorphism. In particular, for $\mathrm{SO}(n)$ and $\mathrm{Spin}(n)$, only the obvious n-dimensional representation is minimal.

Definition **(1.3)** *A group G is strongly unacceptable if there exists a finite group Γ, and element-conjugate homomorphisms $\phi_1, \phi_2 : \Gamma \to G$ such that $\phi_1(\Gamma)$ is not mapped to $\phi_2(\Gamma)$ by any automorphism of G.*

(1.4) Clearly a strongly unacceptable group is unacceptable. An equivalent formulation of strong unacceptability is that there exist element-conjugate homomorphisms $\phi_1, \phi_2 : \Gamma \to G$ such that ϕ_1 is not equal to $S \circ \phi_2 \circ T$ for any automorphisms S and T of G and Γ respectively. Yet another is that there exist element-conjugate homomorphisms $\phi_i : \Gamma \to G$ such that the pairs $(G, \phi_1(\Gamma))$ and $(G, \phi_2(\Gamma))$ are not isomorphic.

§2. Spin Groups

It is proved in [3] that $\mathrm{Spin}(n)$ is acceptable for $n \leq 6$ and unacceptable for $n \geq 9$. We will prove that $\mathrm{Spin}(7)$ is acceptable, $\mathrm{Spin}(8)$ unacceptable, and $\mathrm{Spin}(n)$ *strongly unacceptable* for $n \geq 35$. The last bound is not sharp; with more work, it can certainly be improved to 21, and probably further.

(2.1) Let $n = 2m + 1$ be an odd integer. Let T denote a maximal torus of the spin group $B_m(\mathbb{R}) = \mathrm{Spin}(n)$. The image of T under the covering map $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ is again a maximal torus, which we denote $\bar{T}$. Let $\tilde{T} \cong \mathbb{R}^n$ denote the common universal covering space of T and $\bar{T}$. The character group $X^*(\bar{T}) \cong \mathbb{Z}^m$ has a standard basis $\{e_i\}$ characterized (up to sign) by the property that the non-zero weights of the minimal representation of $\mathrm{SO}(n)$ are $\pm e_i$. This gives a standard dual basis $\{f_i\}$ of the group of cocharacters $X_*(\bar{T})$, and $\bar{T} \cong \mathbb{R}^n / X_*(\bar{T})$. The double cover

$$\mathbb{R}^n / X_*(T) = T \to \bar{T} = \mathbb{R}^n / X_*(\bar{T})$$

corresponds to the index 2 subgroup

$$X_*(\bar{T}) = \{a_1 f_1 + \cdots a_m f_m |\ a_i \in \mathbb{Z},\ a_1 + \cdots + a_m \in 2\mathbb{Z}\}.$$

The Weyl group W of $\mathrm{Spin}(n)$ (resp. $\mathrm{SO}(n)$) with respect to T (resp. $\bar{T}$) is isomorphic to $\mathbb{F}_2^m \rtimes S_m$; the first factor sends $e_i \mapsto \pm e_i$, and the second permutes the e_i. Every element in $\mathrm{SO}(n)$ is conjugate to an element of T unique up to W-action, so $\mathrm{SO}(n)$-conjugacy classes are parametrized by W-orbits in

$$\tilde{T}/X_*(\bar{T}) \cong (\mathbb{R}/\mathbb{Z})^n. \tag{2.1.1}$$

Similarly, $\mathrm{Spin}(n)$ conjugacy classes are parametrized by W-orbits in $\mathbb{R}^n/X_*(T)$.

(2.2) Similar observations apply in the D_m case. As before, let T denote a maximal torus of $\mathrm{Spin}(2m)$ covering a maximal torus $\bar{T}$ of $\mathrm{SO}(2m)$. Let e_i denote a basis for $X^*(\bar{T})$ consisting of weights of the minimal representation of $\mathrm{SO}(2m)$, and f_i a dual basis. The Weyl group of $\mathrm{Spin}(2m)$ (resp. $\mathrm{SO}(2m)$) with respect to T (resp. $\bar{T}$) is isomorphic to $V^m \rtimes S_m$, where V^m is the codimension 1 subspace of $\mathbb{F}_2^m$ orthogonal to the vector $(1, 1, \ldots, 1)$. There exist outer automorphisms of $\mathrm{SO}(2m)$ (which then induce outer automorphisms of $\mathrm{Spin}(2m)$) preserving $\bar{T}$ and acting on it by any given element of

$$\mathbb{F}_2^m \rtimes S_m \setminus V^m \rtimes S_m.$$

The standard embedding $\mathrm{SO}(2m) \to \mathrm{SO}(2m+1)$ sends $\bar{T}$ to a maximal torus of $\mathrm{SO}(2m+1)$, and two elements of $\bar{T}$ are conjugate in $\mathrm{SO}(2m+1)$ if and only if they have the same $\mathbb{F}_2^m \rtimes S_m$-orbit. In other words, two elements of $\mathrm{SO}(2m)$ (resp. $\mathrm{Spin}(2m)$) are conjugate in $\mathrm{SO}(2m+1)$ (resp. $\mathrm{Spin}(2m+1)$) only if there exists an automorphism of $\mathrm{SO}(2m)$ (resp. $\mathrm{Spin}(2m)$) sending one to the other. In general, the orbit of x in $G = \mathrm{SO}(2m)$ (resp. $\mathrm{Spin}(2m)$) under all automorphisms of G (resp. all automorphisms preserving $\mathrm{SO}(2m)$) is strictly smaller than the orbit of x under inner automorphisms. However, if some coordinate of an element of $(\mathbb{R}/\mathbb{Z})^m$ representing x as in (2.1.1) is zero, then the two orbits coincide, because the outer automorphism inverting that coordinate fixes x. This is the case whenever 1 is an eigenvalue of the image of x in the minimal representation.

Lemma **(2.3)** *Let Γ be a finite group and $\chi : \Gamma \to \{\pm 1\}$ a non-trivial character. Let $G = \mathrm{SU}(4) \rtimes \langle T \rangle$, where T denotes complex conjugation. If $\phi_1 : \Gamma \to G$ factors through* $\mathrm{SU}(4)$, *and $\phi_2 := \chi\phi_1$ is element-conjugate to ϕ_1 in G, then ϕ_1 and ϕ_2 are globally conjugate in G.*

Proof. Let ψ_i denote the complex 4-dimensional representation associated to ϕ_i. Then $\psi_1 \oplus \bar{\psi}_1$ and $\psi_2 \oplus \bar{\psi}_2$ are equivalent 8-dimensional representations. Therefore, we can decompose

$$\psi_1 \cong \alpha \oplus \beta,\ \psi_2 \cong \alpha \oplus \bar{\beta},$$

for some pair of representations (α, β). If either α or β is zero-dimensional, then ϕ_1 and ϕ_2 are globally conjugate in G. By hypothesis, there is an equality of multisets

$$\{\chi\alpha_1, \ldots \chi\alpha_m, \chi\beta_1, \ldots, \chi\beta_n\} = \{\alpha_1, \ldots, \alpha_m, \bar{\beta}_1, \ldots, \bar{\beta}_n\}. \tag{2.3.1}$$

We divide the possibilities as follows:

i) $\dim(\beta_1) = 3,\ \dim(\alpha_1) = 1;$

ii) $\dim(\alpha_1) = 3,\ \dim(\beta_1) = 1$;
iii) $\dim(\beta_1) = 2,\ \dim(\alpha_1) = \dim(\beta_2) = 1$;
iv) $\dim(\alpha_1) = 2,\ \dim(\alpha_2) = \dim(\beta_1) = 1$;
v) $\dim(\alpha_1) = \dim(\beta_1) = \dim(\beta_2) = \dim(\beta_3) = 1$;
vi) $\dim(\alpha_1) = \dim(\alpha_2) = \dim(\alpha_3) = \dim(\beta_1) = 1$;
vii) $\dim(\alpha) = \dim(\beta) = 2$.

In case (i), (2.3.1) reads

$$\{\chi\alpha_1, \chi\beta_1\} = \{\alpha_1, \bar{\beta}_1\},$$

so for dimension reasons, $\chi\alpha_1 \cong \alpha_1$. This is impossible because $\dim(\alpha_1) = 1$ and $\chi \neq 1$. In case (ii), we again have $\chi\alpha_1 \cong \alpha_1$, but this time $\dim(\alpha_1) = 3$. Taking determinants, we again deduce $\chi = \chi^3 = 1$. Case (iii) divides into subcases according to whether $\chi\alpha_1$ is matched with α_1 or $\bar{\beta}_2$. The former case is impossible because $\chi \neq 1$, and the latter case implies

$$\alpha_1 \cong \chi\bar{\beta}_2 \cong \bar{\alpha}_1.$$

In other words, $\alpha \cong \bar{\alpha}$, or $\alpha \oplus \bar{\beta} \cong \bar{\alpha} \oplus \bar{\beta}$. As $\bar{\alpha} \oplus \bar{\beta}$ is globally G-conjugate to $\alpha \oplus \beta$, this settles case (iii). Case (iv) divides into two cases, according to whether $\chi\alpha_2 \cong \alpha_2$ (which is impossible) or $\chi\alpha_2 \cong \bar{\beta}_1$ (which implies that $\bar{\beta} \cong \beta$.) Cases (v) and (vi) each divide into seven essentially distinct cases. Each possible matching implies $\alpha = \bar{\alpha}$, $\beta = \bar{\beta}$, or $\chi\gamma = \gamma$ for some character γ.

Case (vii) is the most difficult. It subdivides according to whether $\chi\alpha = \bar{\beta}$ and $\chi\beta = \alpha$, or $\chi\alpha = \alpha$ and $\chi\beta = \bar{\beta}$, or neither. In the first case $\beta = \bar{\beta}$, so ϕ_1 and ϕ_2 are globally $SU(4)$-conjugate. For the second case, we recall that there exists an inner automorphism of $U(2)$ mapping $x \mapsto \bar{x}\det(x)$. Therefore,

$$\chi\beta \cong \bar{\beta} \cong \beta\det(\bar{\beta}),$$

so $\det(\beta)$ (and therefore $\det(\alpha)$) must equal χ. It follows that

$$\bar{\alpha} \oplus \bar{\beta} \cong \chi\alpha \oplus \chi\beta \cong \alpha \oplus \bar{\beta},$$

whence ϕ_1 and ϕ_2 are again G-conjugate. In the last case, α and β must both decompose into 1-dimensional pieces, and the matching (2.3.1) must pair at least one α with another α and at least one α with a β. We may therefore assume $\alpha_2 = \chi\alpha_1$ and $\bar{\beta}_1 = \chi\alpha_2$. We conclude that

$$(\alpha_1, \alpha_2, \beta_1, \beta_2) = \begin{cases} (\alpha_1, \chi\alpha_1, \chi\alpha_1, \alpha),\ \alpha_1^2 = \chi & \text{if } \chi\beta_1 = \alpha_1 \\ (\alpha_1, \chi\alpha_1, \alpha_1, \chi\alpha_1),\ \alpha_1^2 = 1 & \text{if } \chi\beta_1 = \bar{\beta}_2 \end{cases}$$

In either case $\bar{\beta} \cong \beta$,so the lemma holds. □

Proposition **(2.4)** *The group* Spin(7) *is acceptable.*

Proof. Let ϕ_1 and ϕ_2 denote element-conjugate maps from a finite group Γ to Spin(7). Let ψ_i denote the composition of ϕ_i with the covering map Spin(7) $\to$ SO(7). The ψ_i are element-conjugate and therefore, by [3], globally conjugate. Conjugating ϕ_2 if necessary, we may assume without loss of generality that

$$\phi_1(x)\phi_2(x)^{-1}(\gamma) \in \ker(\mathrm{Spin}(7) \to \mathrm{SO}(7)),$$

for all $\gamma \in \Gamma$. The kernel of the covering map is the center $\{1, z\} = Z(\mathrm{Spin}(7))$. Therefore,

$$\Gamma' := \{\gamma \in \Gamma |\ \phi_1(x) = \phi_2(x)\}$$

is a subgroup of Γ of index ≤ 2. If $\Gamma' = \Gamma$, we are done.

Next, let ρ_i denote the composition of ϕ_i with the spin representation ρ of Spin(7). It is well known that ρ is orthogonal. By abuse of notation, we identify ρ (resp. ρ_i) with a homomorphism Spin(7) $\to$ SO(8) (resp. $\Gamma \to$ SO(8)) through which it factors. By [3], O(8) is acceptable, so there exists $g \in$ O(8) such that

$$g\rho_1(\gamma)g^{-1} = \rho_2(\gamma)$$

for all $\gamma \in \Gamma$. In particular, setting

$$K := \rho(\mathrm{Spin}(7)) \cap g^{-1}\rho(\mathrm{Spin}(7))g,$$

we have

$$\rho_1(\Gamma) \subset K(\mathbb{R});\ \ \rho_2(\Gamma) \subset gK(\mathbb{R})g^{-1}.$$

Suppose that $K \cong$ Spin(7). As B_3 has no outer automorphisms, g must act on Spin(7) by an inner automorphism ad(h). This means that ϕ_1 and ϕ_2 are in fact globally conjugate by h. Therefore, it suffices to consider the case that K is a proper subgroup of ρ(Spin(7)).

Note first that as Spin(7) is simply connected, ρ lifts to a map lifts to a map

$$\tilde{\rho}:\ \mathrm{Spin}(7) \to \mathrm{Spin}(8).$$

As ρ is injective, the quotient map

$$\tilde{\rho}(\mathrm{Spin}(7)) \times_{\mathrm{Spin}(8)} \tilde{\rho}(\mathrm{Spin}(7)) \to K$$

is an isomorphism. On the other hand, the triple (Spin(7), $\tilde{\rho}$, Spin(8)) is equivalent under an outer automorphism of D_4 to the triple (Spin(7), $\tilde{\rho}'$, Spin(8)), where $\tilde{\rho}'$ covers the standard embedding $\rho' :$ SO(7) $\to$ SO(8). For any $g \in$ SO(8) which does not normalize ρ'(SO(7)), we have

$$\rho'(\mathrm{SO}(7)) \times_{\mathrm{SO}(8)} g\rho'(\mathrm{SO}(7))g^{-1} = \mathrm{SO}(6)$$

(*cf.* [3]), so if $\tilde{g} \in$ Spin(8) lies over g,

$$\tilde{\rho}'(\mathrm{Spin}(7)) \times_{\mathrm{Spin}(8)} \tilde{g}\tilde{\rho}'(\mathrm{Spin}(7))\tilde{g}^{-1} = \mathrm{Spin}(6).$$

It follows that $K \cong \mathrm{Spin}(6)$. Without loss of generality, therefore, we may assume that

$$\phi_1(\Gamma) \subset \mathrm{Spin}(6) \subset \mathrm{Spin}(7).$$

The commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Spin}(6) & \rightarrow & \mathrm{Spin}(7) \\ \downarrow & & \downarrow \\ \mathrm{SO}(6) & \rightarrow & \mathrm{SO}(7) \end{array}$$

implies that $Z(\mathrm{Spin}(7))$ is contained in the image of $\mathrm{Spin}(6)$. As

$$\phi_2(\Gamma) \subset \phi_1(\Gamma) Z(\mathrm{Spin}(7)),$$

$\phi_2(\Gamma) \subset \mathrm{Spin}(6)$.

Now, $\mathrm{Spin}(6)$ is the same as $\mathrm{SU}(4)$, so we may view $\phi_1(\Gamma)$ and $\phi_2(\Gamma)$ as subgroups of the latter. As $\phi_1(\gamma)$ is conjugate to $\phi_2(\gamma)$ in $\mathrm{Spin}(7)$, there exists an automorphism of $\mathrm{SU}(4)$ sending one to the other. In other words, $\phi_1(\gamma)$ is conjugate to $\phi_2(\gamma)$ in $\mathrm{SU}(4) \rtimes \langle T \rangle$. To apply the lemma above, it suffices to show that

$$\chi(\gamma) := \phi_2(\gamma)\phi_1(\gamma)^{-1}$$

takes values in $\pm 1 \in \mathrm{SU}(4)$. But we know that the image of $\chi(\Gamma)$ in $\mathrm{Spin}(7)$ lies in the center of $\mathrm{Spin}(7)$, or in other words, that the image of $\chi(\Gamma)$ in $\mathrm{SO}(7)$ is the identity. The composed homomorphism $\mathrm{SU}(4) \to \mathrm{SO}(7) \to \mathrm{GL}(7)$ is associated with the direct sum of the exterior square of the minimal representation of $\mathrm{SU}(4)$ and the trivial representation. The kernel of the exterior square representation is clearly ± 1. The proposition follows. □

Proposition **(2.5)** *The group* $\mathrm{Spin}(8)$ *is unacceptable.*

Proof. Let $\Gamma = \mathrm{SL}(3, \mathbb{F}_2)$ and ρ denote the unique irreducible 8-dimensional representation of Γ (the Steinberg representation). This representation is orthogonal and unimodular, so we may take it to factor through $\psi_1 : \Gamma \to \mathrm{SO}(8)$. Let M denote any element of $\mathrm{O}(8)$ of determinant -1, and let

$$\psi_2(\gamma) = M \psi_1(\gamma) M^{-1}$$

for all $\gamma \in \Gamma$. In [3] it is shown that ψ_1 and ψ_2 are element-conjugate but not globally conjugate in $\mathrm{SO}(8)$. Let

$$\tilde{\Gamma} := \Gamma \times_{\mathrm{SO}(8)} \mathrm{Spin}(8).$$

The projection $\tilde{\Gamma} \to \mathrm{Spin}(8)$ is injective. Let x denote an element of Γ of order 2. By [3], the eigenvalues 1 and -1 of $\rho(x)$ each occur with multiplicity 4. Letting $\bar{T}$ denote a maximal torus of $\mathrm{SO}(8)$ containing $\psi_1(x)$, we see that x corresponds to the W-orbit of $\left(\frac{1}{2}, \frac{1}{2}, 0, 0\right) \in (\mathbb{R}/\mathbb{Z})^4$. Thus $\psi_1(x)$ lifts to an element of order 2 in $\mathrm{Spin}(8)$. Both elements over x in the unique non-trivial double cover of Γ have order 4. We conclude that $\tilde{\Gamma} = \Gamma \times \mathbb{Z}/2\mathbb{Z}$, and therefore that ψ_1 lifts to a homomorphism $\phi_1 : \Gamma \to \mathrm{Spin}(8)$. Let ϕ_2 denote the composition of ψ_1 with an outer automorphism of $\mathrm{Spin}(8)$ lifting conjugation by M. If ϕ_1 and ϕ_2 were globally $\mathrm{Spin}(8)$-conjugate, their compositions with $\mathrm{Spin}(8) \to \mathrm{SO}(8)$, ψ_1 and ψ_2, would likewise be globally conjugate, contrary to [3]. It remains only to prove that $\phi_1(\gamma)$ is conjugate to $\phi_2(\gamma)$ for all $\gamma \in \Gamma$. This is immediate from the conjugacy of $\psi_1(\gamma)$ and $\psi_2(\gamma)$ in the case that γ is of odd order. When γ is of even order, it follows from (2.2) and the fact [2] that 1 occurs as an eigenvalue of $\rho(\gamma)$. □

Proposition **(2.6)** *For all $n \geq 35$,* Spin(n) *is strongly unacceptable.*

Proof. Let ρ denote the 35-dimensional representation of $\Gamma = S_{10}$ associated with the partition $10 = 2+2+1+\cdots+1$. By [2] pp. 49, 237, ρ factors through a homomorphism $\psi : \Gamma \to \mathrm{SO}(35)$. If $x \in \Gamma$ denotes a transposition, then $\mathrm{tr}(\rho(x)) = -21$, so $\rho(x)$ has eigenvalues 1 and -1 with multiplicities 28 and 7 respectively. By (2.1), $\psi(x)$ corresponds to the W-orbit in $(\mathbb{R}/\mathbb{Z})^{17}$ containing

$$(\underbrace{\frac{1}{2},\ldots,\frac{1}{2}}_{14},0,0,0).$$

Therefore, $\psi(x)$ lifts to an element of order 2 in Spin(35). The only double cover of Γ in which x is covered by an element of order 2 is $\Gamma \times \mathbb{Z}/2\mathbb{Z}$. Therefore, $\psi(x)$ lifts to a homomorphism $\phi_1 : \Gamma \to \mathrm{Spin}(35)$. Let

$$\phi_2(\gamma) = \phi_1(\gamma)\chi(\gamma),$$

where χ maps Γ onto the center of Spin(35) with kernel $\mathcal{A}_{10}$. As ρ is irreducible, the centralizer of $\psi(\Gamma)$ is

$$Z(\mathrm{SO}(35)) = \{1\},$$

so if

$$g\phi_1(\gamma)g^{-1} = \phi_2(\gamma)$$

for all $\Gamma \in \Gamma$, then g lies in the center of Spin(35), and therefore ϕ_1 and ϕ_2 are equal, which is absurd.

We claim that ϕ_1 and ϕ_2 are element-conjugate. By [3], it suffices to prove that -1 is an eigenvalue of $\psi(x)$ for all $x \in \Gamma \setminus \mathcal{A}_{10}$. Let sgn denote the sign character and σ the permutation representation of Γ. Then

$$\mathrm{Sym}^2\sigma = \mathrm{sgn} \otimes \rho \oplus \sigma \oplus \sigma.$$

Let $m(M)$ denote the multiplicity of the eigenvalue 1 for M. If $a = m(\sigma(x))$, $b = m(-\sigma(x))$, we have

$$\begin{aligned} m\left(\mathrm{Sym}^2\sigma(x)\right) &\geq \binom{a+1}{2} + \binom{b+1}{2} + \frac{10-a-b}{2} \\ &= \frac{a^2+b^2+10}{2} = \frac{(a-2)^2+b^2}{2} + 2a+3 \geq 2a+3 > 2m(\sigma(x)). \end{aligned}$$

Therefore, ϕ_1 and ϕ_2 are indeed element-conjugate. However, both S_{10} and Spin(35) have only inner automorphisms, so Spin(35) must in fact be strongly unacceptable.

For $n \geq 35$, we keep $\Gamma = S_{10}$ and define ϕ'_i to be the composition of ϕ_i with the standard embedding $i :$ Spin$(35) \hookrightarrow$ Spin(n). Evidently, ϕ'_1 and ϕ'_2 are element-conjugate. We claim they are not globally Spin(n)-conjugate. Indeed, if

$$g\phi'_1(\gamma)g^{-1} = \phi'_2(\gamma)$$

for all $\gamma \in \Gamma$, then

$$\Gamma \subset i(\mathrm{Spin}(35)) \cap g^{-1} i(\mathrm{Spin}(35)) g \subset \mathrm{Spin}(n).$$

However, this intersection is a double cover of

$$\bar{i}(\mathrm{SO}(35)) \cap \bar{g}^{-1} \bar{i}(\mathrm{SO}(35)) \bar{g} \subset \mathrm{SO}(n). \tag{2.6.1}$$

Now $i(\mathrm{SO}(35))$ is the pointwise stabilizer in $\mathrm{SO}(n)$ of a subspace W of codimension 35 in the minimal representation space. Therefore, the intersection (2.6.1) is the pointwise stabilizer of $W + \bar{g}^{-1} W$. In particular, (2.6.1) is of the form $\mathrm{SO}(m)$. This is impossible if $m < 35$ since the original representation ρ is irreducible. It follows that $\bar{g}$ normalizes $\mathrm{SO}(35)$, which means that g normalizes $\mathrm{Spin}(35)$. On the other hand, $\mathrm{Spin}(35)$ has no inner automorphisms, so g acts on $\mathrm{Spin}(35)$ by $\mathrm{ad}(h)$ for some $h \in \mathrm{Spin}(35)$. This is impossible since ϕ_1 and ϕ_2 are not globally conjugate. It follows that $\mathrm{Spin}(n)$ is unacceptable for $n \geq 35$. For $n \geq 35$ odd, $G = \mathrm{Spin}(n)$ is strongly unacceptable since all automorphisms of G and Γ are inner. For n even and greater than 8, all automorphisms of $\mathrm{Spin}(n)$ preserve $\ker(\mathrm{Spin}(n) \to \mathrm{SO}(n))$, so every automorphism of $\mathrm{Spin}(n)$ is obtained by restriction from an inner automorphism of $\mathrm{Spin}(n+1)$. In particular, since $\mathrm{Spin}(n+1)$ is strongly unacceptable, the composition $\phi'_2 \circ T$ cannot be equal to ϕ'_1. □

Theorem **(2.7)** *If G is a connected, simply connected, compact Lie group which is strongly unacceptable, then there exist non-isometric isospectral manifolds X_1 and X_2 with universal covering space G. In particular, such X_i exist for $G = \mathrm{Spin}(n)$ for $n \geq 35$.*

Proof. Let $\phi_i : \Gamma \to G$ denote element-conjugate homomorphisms such that $(G, \phi_1(\Gamma))$ and $(G, \phi_2(\Gamma))$ are not isomorphic. Let $X_i = G/\phi_i(\Gamma)$. By the elementary trace formula [5] Lemma 1, the spectra of X_i depend only on the conjugacy classes of $\phi_i(\gamma)$ and are therefore independent of i. It remains to show that X_1 and X_2 are not isometric.

The image of $1 \in G$ in X_i gives X_i the structure of pointed space. As X_1 and X_2 are homogeneous spaces, any isometry ψ between them can be chosen to preserve base points. We can therefore lift ψ to an isometry

$$\tilde{\psi} : G = \tilde{X}_1 \to \tilde{X}_2 = G$$

which preserves the identity and makes the diagram

$$\begin{array}{ccc} G & \xrightarrow{\tilde{\psi}} & G \\ \downarrow & & \downarrow \\ X_1 & \xrightarrow{\psi} & X_2 \end{array} \tag{2.7.1}$$

commute. As G is simple and its metric is bi-invariant, we may regard it as as irreducible symmetric space $G \times G/\Delta(G)$, where

$$\Delta(x) = (x, x^{-1}),$$

and the symmetry map exchanges coordinates. By a computation of É. Cartan [6] 8.8.1, all isometries of G come from group automorphisms or anti-automorphisms, so $\tilde{\psi} \in \mathrm{Aut}(G)$ or $\tilde{\psi} \circ \iota \in \mathrm{Aut}(G)$, where ι denotes the map $x \mapsto x^{-1}$. By (2.7.1),

$$\tilde{\psi}(\iota(\phi_1(\Gamma))) = \tilde{\psi}(\phi_1(\Gamma))$$

lies over the base point in X_2. In other words, $\psi(\phi_1(\Gamma)) \subset \phi_2(\Gamma)$. By symmetry, ψ is gives isomorphism between $(G, \phi_1(\Gamma))$ and $(G, \phi_2(\Gamma))$, contrary to assumption. □

(2.8) We conclude the treatment of spin groups by recalling for future reference the proof [3] that Spin(9) is unacceptable. Let $\bar{\Gamma}$ denote the Mathieu group M_{10}. It is known [2] that $\bar{\Gamma}$ is an extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathcal{A}_6$. Consider the composition $\bar{\Gamma} \to \mathrm{SO}(9)$ of the 9-dimensional irreducible representation of the alternating group $\mathcal{A}_{10}$ with the inclusion $\bar{\Gamma} \to \mathcal{A}_{10}$. Define

$$\Gamma := \bar{\Gamma} \times_{\mathrm{SO}(9)} \mathrm{Spin}(9),$$

and let $\phi_1 : \Gamma \to \mathrm{Spin}(9)$ denote projection into the second factor. Let χ denote the composition

$$\Gamma \twoheadrightarrow \bar{\Gamma} \twoheadrightarrow \mathbb{Z}/2Z \hookrightarrow Z(\mathrm{Spin}(9)) \subset \mathrm{Spin}(9),$$

and let $\phi_2 := \phi_1\chi$. Then ϕ_1 and ϕ_2 are element-conjugate but not globally conjugate. Moreover, the composition with ϕ_i with the 9-dimensional representation of Spin(9) is irreducible, so by the argument of Prop. (2.6), Spin(n) is unacceptable for $n \geq 9$.

(2.9) It is worth noting that $\mathcal{A}_6$ is a subgroup of Γ. Indeed, $\mathcal{A}_6$ is an index 2 subgroup of M_{10}, so some double cover of $\mathcal{A}_6$ lies in Γ. There is a unique non-trivial double cover $\widetilde{\mathcal{A}}_6$ of $\mathcal{A}_6$, and any element $x \in \widetilde{\mathcal{A}}_6$ lying over a product of two disjoint transpositions is of order 4. On the other hand, the image of $\phi_i(x)$ under the 9-dimensional representation of Spin(9) has eigenvalues 1 and -1 with multiplicities 5 and 4 respectively, so $\phi_i(x)$ is of order 2. Therefore,

$$\Gamma \supset \mathcal{A}_6 \times \mathbb{Z}/2\mathbb{Z} \supset \mathcal{A}_6. \tag{2.9.1}$$

§3. Exceptional Groups

It is known [3] that $G_2(\mathbb{R})$ is acceptable and $F_4(\mathbb{R})$ unacceptable. In this section, we prove that $E_6(\mathbb{R})$, $E_6^{\mathrm{ad}}(\mathbb{R})$, $E_7(\mathbb{R})$, $E_7^{\mathrm{ad}}(\mathbb{R})$, and $E_8(\mathbb{R})$ are all unacceptable. The group Γ in all of these cases is the double cover of M_{10} constructed in section (2.8). Throughout this section, it will always be denoted Γ and ϕ_1 and ϕ_2 will denote the homomorphisms $\Gamma \to \mathrm{Spin}(9)$ constructed above. We begin with the simply connected cases.

(3.1) We fix once and for all a chain of inclusions

$$\mathrm{Spin}(9) = B_4(\mathbb{R}) \subset E_6(\mathbb{R}) \subset E_7(\mathbb{R}) \subset E_8(\mathbb{R}). \tag{3.1.1}$$

Let ψ_i denote the composition of ϕ_i with the chain of homomorphisms. Let $\mathcal{A}_6$ denote the index 4 subgroup of Γ in (2.9.1), and ρ the restriction of the adjoint representation of $E_8(\mathbb{R})$ to $\mathcal{A}_6$ by ψ_i. Note that ρ does not depend on i. It can be obtained by first restricting Lie(E_8) to $B_4(\mathbb{R})$ and then restricting to $\mathcal{A}_6 \subset \Gamma$ by ϕ_i. By [4], the restriction of the adjoint representation of E_8 to B_4 is $V_{36} + 8V_{16} + 7V_9 + 21V_1$, where V_1, V_9, V_{16}, and V_{36} denote the trivial, minimal, spin, and adjoint representations respectively.

(3.2) From the character table of $\mathcal{A}_6$ is it easy to see that V_1 and V_9 are irreducible, V_{16} the sum of two irreducible 8-dimensional representations, and V_{36} the sum of two irreducible 8-dimensional representations and one 10-dimensional representation W_{10} with multiplicity 2. Therefore, ρ contains W_{10} with multiplicity 2. Both copies of W_{10} are contained in the image of $\mathrm{Lie}(B_4)$ in the representation space $\mathrm{Lie}(E_8)$ of ρ. If

$$g\phi_1(\gamma)g^{-1} = \phi_2(\gamma)$$

for all $\gamma \in \mathcal{A}_6$, then gB_4g^{-1} also contains both copies of W_{10}. Therefore

$$\dim_{\mathbb{C}}(\mathrm{Lie}(B_4) \cap g\mathrm{Lie}(B_4)g^{-1}) \geq 20.$$

Setting

$$K := B_4 \cap gB_4g^{-1},$$

we see that $\dim(K) \geq 20$. On the other hand,

$$\psi_2(\Gamma) \subset K(\mathbb{R}).$$

The composition of ψ_2 with the 9-dimensional representation of Spin(9) is irreducible, so $K(\mathbb{R})$ admits an irreducible 9-dimensional representation. Let H denote the derived group of the identity component of K. Its dimension is at least 16 since the $\mathrm{rank}(B_4) = 4$. On the other hand, H is normal in $B_4 \cap B_4g^{-1}$, so the restriction of the minimal representation of Spin(9) to H is a direct sum of irreducible representations of equal dimension. From the classification of semisimple subalgebras of $\mathrm{Lie}(B_4)$ (see *e.g.* [4]), there is no proper subalgebra of dimension ≥ 16 with an irreducible representation of dimension dividing 9. Therefore, $H = B_4$, and g normalizes $B_4(\mathbb{R})$.

As B_4 has no outer automorphisms, the action of g on $B_4(\mathbb{R})$ is that of $\mathrm{ad}(h)$ for some $h \in B_4(\mathbb{R})$. It follows that

$$h\psi_1(\gamma)h^{-1} = \psi_2(\gamma)$$

for all $\gamma \in \Gamma$. We have already seen that this is possible only for h in the center of $B_4(\mathbb{R})$. We conclude that $E_8(\mathbb{R})$ is unacceptable. If ϕ_1 and ϕ_2 are not globally $E_8(\mathbb{R})$-conjugate, *a fortiori*, they are not globally $E_7(\mathbb{R})$-conjugate, $E_6(\mathbb{R})$-conjugate, or $F_4(\mathbb{R})$-conjugate.

(3.3) The argument above works equally well with E_8 replaced by E_7^{ad} or E_6^{ad}. One has only to replace the chain of inclusions (3.1.1) with

$$B_4(\mathbb{R}) \subset E_6(\mathbb{R}) \subset E_7^{\mathrm{ad}}(\mathbb{R})$$

and

$$B_4(\mathbb{R}) \subset E_6^{\mathrm{ad}}(\mathbb{R}).$$

The only slightly non-trivial point is the fact that $E_6 \to E_7^{\mathrm{ad}}$ is an inclusion. This can be checked by restricting the adjoint representation of E_7 to E_6 and observing that it contains the (faithful) minimal representations of E_6. In each case, the restriction of the adjoint representation of the largest group in the chain to $\mathcal{A}_6$ contains exactly two copies of W_{10}. We therefore have the following theorem:

Theorem **(3.4)** *The only acceptable, compact, exceptional, simple Lie group is* $G_2(\mathbb{R})$

Proof. The acceptability of G_2 is proved in [3]. All the rest has been shown above. □

Corollary **(3.5)** *The only acceptable, complex, exceptional, simple Lie group is* $G_2(\mathbb{C})$.

Proof. Immediate from the above theorem and [3].

Theorem **(3.6)** *A connected, simply connected, compact Lie group is acceptable if and only if it has no direct factors of the form* $B_n(\mathbb{R})$ $(n \geq 4)$, $D_n(\mathbb{R})$ $(n \geq 4)$, E_n, *or* F_4.

Proof. Every connected, simply connected compact Lie group is of the form $G(\mathbb{R})$, where G is a simply connected, semisimple algebraic group. In particular, G is a product of simple algebraic groups G_i, and

$$G(\mathbb{R}) = \prod_i G_i(\mathbb{R}).$$

By [3], $G(\mathbb{R})$ is acceptable if and only if all $G_i(\mathbb{R})$ are so. Furthermore, in [3] it is shown that $B_2(\mathbb{R})$ and $G_2(\mathbb{R})$ are acceptable and $A_n(\mathbb{R})$ and $C_n(\mathbb{R})$ are acceptable for all n. All other cases are treated above. □

Corollary **(3.7)** *If* G *is a connected, simply connected, semisimple algebraic group over* $\mathbb{C}$, *then* $G(\mathbb{C})$ *is acceptable if and only if it has no direct factors of the form* $\mathrm{Spin}(n, \mathbb{C})$ $(n \geq 8)$, $E_n(\mathbb{C})$, *or* $F_4(\mathbb{C})$.

Proof. Immediate from the above theorem and [3]. □

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