THE FRACTIONAL CHROMATIC NUMBER OF A GRAPH AND A CONSTRUCTION OF MYCIELSKI

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ABSTRACT

The fractional chromatic number of a graph G lies between the clique number and the chromatic number of G. In many familiar examples the fractional chromatic number is within one unit of the clique number. Here we show that this is not always the case by constructing a sequence of graphs with clique number 2 and with fractional chromatic number given by the sequence (a_n) with $a_0 = 1$ and $a_{n+1} = a_n + (1/a_n)$.

INTRODUCTION

All our graphs are finite and simple. We write $V(G)$ for the vertex set of G and $E(G)$ for the edge set of G. We denote by $\chi(G)$ the chromatic number of the graph G, i.e. the least number of colors that can be assigned to the vertices of G in such a way that adjacent vertices are assigned different colors. A homomorphism from a graph G to a graph H is a map $\phi : V(G) \to V(H)$ such that $uv \in E(G)$ implies $\phi(u)\phi(v) \in E(H)$. It is well-known and easy to see that $\chi(G) \leq r$ if and only if there is a graph homomorphism from G to K_r, the complete graph on r vertices.

We write $\chi_F(G)$ for the fractional chromatic number (or set-chromatic number [2], or ultimate chromatic number [3], or multi-coloring number [4]) of G, which is defined as follows. Say that a graph has an a/b-coloring if, to each vertex of G, one can assign a b-element subset of $\{1, 2, 3, \ldots, a\}$ in such a way that adjacent vertices are assigned disjoint subsets. Define $\chi_F(G) = \inf\{\frac{a}{b} : G \text{ can be } a/b\text{-colored}\}$. G can be a/b-colored if and only if there is a graph homomorphism from G to the Kneser graph [13], here denoted by $K_{a/b}$, which is the graph whose vertices are the b-element subsets of a fixed a-element set and with an edge between two vertices if and only if they are disjoint as sets. An example of a $5/2$-coloring of the pentagon C_5 is given in Figure 1.

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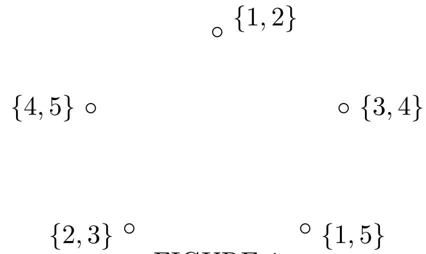

FIGURE 1

Every color class in this 5/2-coloring has $\alpha(G)$ elements, where $\alpha(G)$ is the independence number of G, the cardinality of the largest independent set of vertices in G. This implies that the coloring is optimal, i.e., that $\chi_F(C_5) = \frac{5}{2}$. (More generally, $\chi_F(G)$ is bounded below by $\frac{|V(G)|}{\alpha(G)}$.)

We use the slash in the notations "a/b-coloring" and "$K_{a/b}$ to suggest the fraction $\frac{a}{b}$; however, a/b must be viewed as a formal symbol, since a graph may be a/b-colorable but not c/d-colorable even if $\frac{a}{b} = \frac{c}{d}$. As an example, the Kneser graph $K_{6/2}$ is 6/2-colorable but not 3/1-colorable (i.e., not 3-colorable).

The parameter $\chi(G)$ is well-known to be the optimal solution to the following integer program. Number the vertices of G from 1 to m and the maximal independent sets of vertices of G from 1 to n. Define the $m \times n$ matrix A by $a_{ij} = 1$ if the ith vertex is in the jth independent set, and 0 otherwise. Then the problem of computing $\chi(G)$ can be expressed as:

$$\begin{array}{ll} \text{minimize} & \vec{x} \cdot \vec{1} \\ \text{subject to} & A\vec{x} \geq \vec{1} \\ \text{with} & \vec{x} \in \mathbf{Z}^n, \quad \vec{x} \geq \vec{0}, \end{array}$$

where we write $\vec{1}$ for the vector each of whose entries is 1. It is not hard to see that, if we relax the requirement that $\vec{x}$ have integer components in this program, the resulting linear program computes the fractional chromatic number $\chi_F(G)$. Taking this viewpoint, and using standard facts about the region of feasible solutions to a linear program, one learns that the infimum in the definition of $Chif(G)$ is always achieved and that $Chif(G)$ is always rational. In fact, $\chi_F(K_{a/b}) = \frac{a}{b}$, so every rational number greater than or equal to 2 is $\chi_F(G)$ for some graph G.

Recall that a clique in a graph is a set of mutually adjacent vertices, or, alternatively, a set of vertices no two of which lie in any independent set. The integer program which calculates the clique number $\omega(G)$ of a graph G is

$$\begin{array}{ll} \text{maximize} & \vec{u} \cdot \vec{1} \\ \text{subject to} & A^{\mathbf{T}}\vec{u} \leq \vec{1} \\ \text{with} & \vec{u} \in \mathbf{Z}^r, \quad \vec{u} \geq \vec{0}, \end{array}$$

where A is the same matrix as the one defined above. This is the dual (in the sense of mathematical programming) of the integer program which calculates $\chi(G)$. When we remove the restriction that the components of $\vec{u}$ be integers, the resulting linear program calculates what is called the fractional clique number $\omega_F(G)$ of the graph G.

Here is a combinatorial interpretation of $\omega_F(G)$. Call a multiset of vertices S an a/b-clique if the cardinality of S (counted with multiplicity) equals a and the number of vertices (again counted with multiplicity) in S that lie in any fixed independent set is at most b. (As an example, the entire vertex set of the pentagon C_5 is a 5/2-clique.) Then $\omega_F(G) = \sup\{\frac{a}{b} : G \text{ has an } a/b\text{-clique}\}$. Here is a rescaled version of this notion that we use in what follows. For any graph G, a fractional clique is a map $f : V(G) \to [0,1]$ such that, if S is any independent set of vertices in $V(G)$, $\sum_{v \in S} f(v) \leq 1$. The fractional clique number $\omega_F(G)$ is equal to $\sup\{\sum_{v \in V(G)} f(v)\}$, where the supremum is taken over all fractional cliques f.

¿From this programming point of view we learn that $\omega(G) \leq \omega_F(G) = \chi_F(G) \leq \chi(G)$ for all graphs G. The equality here follows from the duality theorem of linear programming.

Although there is a duality between ω and χ, there is a certain lack of symmetry as well. The parameter $\chi_F(G) = \omega_F(G)$ need not be at the midpoint of the interval $[\omega(G), \chi(G)]$ and seems to favor the lower endpoint. For example, if G is the Kneser graph $K_{a/b}$, then $\omega(G) = \lfloor \frac{a}{b} \rfloor$, $\chi_F(G) = \omega_F(G) = \frac{a}{b}$, while $\chi(G) = a - 2b + 2$ [6]. For another example, choose $n \geq 2$ and take G to be C_{2n+1}, the cycle on $2n+1$ vertices; then $\omega(G) = 2$, $\chi_F(G) = \omega_F(G) = \frac{2n+1}{n}$, while $\chi(G) = 3$. Another surprising asymmetry is the following: $\chi_F(G)$ is always equal to $\lim_{n\to\infty} \sqrt[n]{\chi(G^n)}$, where the power of G is relative to either the disjunctive or lexicographic product of graphs (see [3], [8], [11]); on the other hand, it is not true that $\omega_F(G)$ always equals $\lim_{n\to\infty} \sqrt[n]{\omega(G^n)}$. In fact, this limit gives the Shannon capacity of the complement of G [12], which is known to be $\sqrt{5}$ when G is the pentagon C_5 [7] (while $\omega_F(C_5) = \frac{5}{2}$).

We do not know if $\chi_F = \omega_F$ can ever be closer to χ than it is to ω. The main theorem of this note, however, shows that the difference between $\chi_F = \omega_F$ and ω can be large.

THE GRAPH TRANSFORMATION OF MYCIELSKI

Motivated by [9], given a graph G we define a graph $\mu(G)$ as follows. If G has vertex set $\{v_1, v_2, \ldots, v_m\}$, let $V(\mu(G)) = \{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m, z\}$ with $x_i x_j \in E(\mu(G))$ if and only if $v_i v_j \in E(G)$, $x_i y_j \in E(\mu(G))$ if and only if $v_i v_j \in E(G)$, $y_i z \in E(\mu(G))$ for all i from 1 to m, and no other edges.

Theorem. *Suppose that G has at least one edge. Then*

$$
\begin{aligned}
&a) \quad \omega(\mu(G)) = \omega(G); \\
&b) \quad \chi(\mu(G)) = \chi(G) + 1; \\
\textit{and} \quad &c) \quad \chi_F(\mu(G)) = \chi_F(G) + \frac{1}{\chi_F(G)}.
\end{aligned}
$$

Proof: a) Since G is an induced subgraph of $\mu(G)$, $\omega(G) \leq \omega(\mu(G))$. To see the opposite inequality, note that the vertex z is in no cliques of size bigger than 2. Also, were $\{x_{i(1)}, x_{i(2)}, \ldots, x_{i(r)}, y_{j(1)}, y_{j(2)}, \ldots, y_{j(s)}\}$ a clique in $\mu(G)$, then the sets $\{i(1), \ldots, i(r)\}$ and $\{j(1), \ldots, j(s)\}$ would be disjoint and $\{v_{i(1)}, v_{i(2)}, \ldots, v_{i(r)}, v_{j(1)}, v_{j(2)}, \ldots, v_{j(s)}\}$ would be a clique. Hence $\omega(G) \geq \omega(\mu(G))$ as well.

b) If $k : V(G) \to \{1, 2, \ldots, n\}$ is a proper coloring of G, then we define a coloring $h : V(\mu(G)) \to \{1, 2, \ldots, n, n+1\}$ of $\mu(G)$ by setting $h(x_i) = h(y_i) = k(v_i)$ for all i, and $h(z) = n+1$. This is easily seen to be a proper coloring of $\mu(G)$ and uses only one more color than the coloring of G. Hence $\chi(\mu(G)) \leq \chi(G)+1$. On the other hand, if h is any proper coloring of $\mu(G)$, we define a coloring k of G by

$$
k(v_i) = \begin{cases} h(x_i) & \text{if } h(x_i) \neq g(z); \\ h(y_i) & \text{if } h(x_i) = h(z). \end{cases}
$$

This is a proper coloring of G which does not use the color $h(z)$. So G can be colored with fewer colors than are required to color $\mu(G)$. Hence $\chi(\mu(G)) \geq \chi(G) + 1$ as well.

c) First, suppose $\chi_F(G) = \frac{a}{b}$ and we have a proper a/b-coloring of G. We produce an $(a^2+b^2)/(ab)$-coloring of $\mu(G)$ as follows. Imagine that each of the a colors has a offspring, b male and $a-b$ female. Color x_i with all the offspring of the colors that are associated to v_i. Color y_i with all the female offspring of the colors that are associated with v_i and with wholly new colors $\{c_1, c_2, \ldots, c_{b^2}\}$. Color z with all the male offspring of all the original colors. Note that this set coloring is proper. There are a^2 offspring colors and b^2 new colors, making a^2+b^2 all told. The resulting coloring of $\mu(G)$ assigns exactly ab colors to each vertex. Hence $\chi_F(\mu(G)) \leq \chi_F(G) + (\chi_F(G))^{-1}$.

To prove the opposite inequality, suppose f is a fractional clique on G that achieves $\omega_F(G)$. We define a map $g : V(\mu(G)) \to [0,1]$ as follows.

$$\begin{aligned} g(x_i) &= \left(1 - \frac{1}{\omega_F(G)}\right) f(v_i), \\ g(y_i) &= \frac{1}{\omega_F(G)} f(v_i), \\ g(z) &= \frac{1}{\omega_F(G)}. \end{aligned}$$

We now show that g is a fractional clique on $\mu(G)$.

If $M \subset V(G)$, let $x(M) = \{x_i : v_i \in M\}$ and let $y(M) = \{y_i : v_i \in M\}$. Let S be an independent set in $\mu(G)$. If $z \in S$, then $S = \{z\} \cup x(M)$ for some independent set $M \subset V(G)$, in which case

$$\sum_{v\in S} g(v) = \frac{1}{\omega_F(G)} + \left(1 - \frac{1}{\omega_F(G)}\right) \sum_{v\in M} f(v) \leq \frac{1}{\omega_F(G)} + \left(1 - \frac{1}{\omega_F(G)}\right) = 1.$$

If $z \notin S$, then $S = x(M) \cup y(N)$ for some (independent) set $M \subset V(G)$ and some $N \subset V(G)$. Because S is an independent set, N may be partitioned into a subset A of M and a set B of vertices which are neither elements of M nor adjacent to elements of M. Then

$$\begin{aligned} \sum_{v\in S} g(v) &= \left(1 - \frac{1}{\omega_F(G)}\right) \sum_{v\in M} f(v) + \frac{1}{\omega_F(G)} \sum_{v\in N} f(v) \\ &= \left(1 - \frac{1}{\omega_F(G)}\right) \sum_{v\in M} f(v) + \frac{1}{\omega_F(G)} \sum_{v\in A} f(v) + \frac{1}{\omega_F(G)} \sum_{v\in B} f(v) \\ &\leq \left(1 - \frac{1}{\omega_F(G)}\right) \sum_{v\in M} f(v) + \frac{1}{\omega_F(G)} \sum_{v\in M} f(v) + \frac{1}{\omega_F(G)} \sum_{v\in B} f(v) \\ &= \sum_{v\in M} f(v) + \frac{1}{\omega_F(G)} \sum_{v\in B} f(v). \qquad (*) \end{aligned}$$

Let H be the subgraph of G induced on B. Say that $\omega_F(H) = \chi_F(H) = \frac{a}{b}$ and that we have an a/b-coloring of H. Then the a color classes $C_1, C_2, \ldots, C_a$ are independent sets in H, and the sets of the form $M \cup C_i$ are independent sets in G. Because f is a fractional clique,

$$\sum_{v\in M} f(v) + \sum_{v\in C_i} f(v) \leq 1$$

for all i. Adding these a inequalities gives

$$a \sum_{v\in M} f(v) + b \sum_{v\in B} f(v) \leq a.$$

Dividing through by a yields

$$\sum_{v\in M} f(v) + \frac{1}{\omega_F(H)} \sum_{v\in B} f(v) \leq 1.$$

Since $\omega_F(G) \geq \omega_F(H)$, $(*)$ above is less than or equal to 1, and so g is a fractional clique.

It follows that

$$\begin{aligned}
\chi_F(\mu(G)) &= \omega_F(\mu(G)) \\
&\geq \sum_{v\in V(\mu(G))} g(v) \\
&= \left(1-\frac{1}{\omega_F(G)}\right)\sum_{v\in V(G)} f(v) + \frac{1}{\omega_F(G)}\sum_{v\in V(G)} f(v) + \frac{1}{\omega_F(G)} \\
&= \sum_{v\in V(G)} f(v) + \frac{1}{\omega_F(G)} \\
&= \omega_F(G) + \frac{1}{\omega_F(G)} \\
&= \chi_F(G) + \frac{1}{\chi_F(G)}
\end{aligned}$$

and the theorem is proved. □

Let G_2 be K_2, the complete graph on two vertices, and recursively define $G_{n+1} = \mu(G_n)$ for $n \geq 2$. This definition makes G_3 the pentagon and G_4 the so-called Grötzsch graph. Our theorem shows that G_n is triangle-free yet has chromatic number n. (This much was known to Mycielski in 1955 [9].) Our theorem also shows that the fractional chromatic number of G_n equals a_n, where $a_2 = 2$ and $a_{n+1} = a_n + a_n^{-1}$. It is known that this sequence grows like $\sqrt{2n}$ in the sense that $a_n/\sqrt{2n} \to 1$ as $n \to \infty$. (See [5], p. 49, [10], problem 60, or [1], problem E3276 for more detailed information about the growth of this sequence.) Hence $\chi(G_n) - \chi_F(G_n) \to \infty$ and $\chi_F(G_n) - \omega(G_n) \to \infty$.

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