TOPOLOGICAL HOCHSCHILD HOMOLOGY AND THE CONDITION OF HOCHSCHILD-KOSTANT-ROSENBERG

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0. INTRODUCTION

The A-module structure of the topological Hochschild homology of smooth algebras A over $k = \mathbb{Z}$ and $k = \mathbb{Z}/p$ was first calculated by Pirashvili in [11] as Mac Lane homology, which is the same as topological Hochschild homology by [12]. In this paper we extend and generalize Pirashvili's result. First, our computation gives the multiplicative structure of the ring $\mathrm{THH}_*(A)$. This is fairly straightforward in the absence of 2-torsion but requires some work in general. Second, we treat a wider class of rings than [11]. Our class is given by a homological criterion described below; it includes several interesting rings for which the topological Hochschild homology was hitherto unknown, including the ring $\mathbb{Z}_p$ of p-adic integers and the power series ring $\mathbb{Z}/p[\![t]\!]$. We also deal with algebras over base rings other than $\mathbb{Z}$ and $\mathbb{Z}/p$.

Recall that if k is a commutative ring, and A is a commutative k-algebra, there is a natural map on Hochschild homology relative to k:

$$\lambda_*^{A/k}\colon \bigwedge\nolimits^* \mathrm{HH}_1^k(A) \to \mathrm{HH}_*^k(A).$$

According to a celebrated theorem of Hochschild, Kostant, and Rosenberg, the map λ_* is an isomorphism whenever A is smooth over k. More generally, we say that a k-algebra A is *H-smooth* if it behaves like a smooth algebra from the homological point of view: if $\mathrm{HH}_*(A)$ is flat over k and $\lambda_*^{A/k}$ is an isomorphism in all dimensions. In the case of algebras of finite type, this implies smoothness. But we are mainly interested in algebras which are not of finite type over k, in particular, completions of affine k-algebras.

Our main result is:

THEOREM 3.2: **(i)** *Let A be an H-smooth k-algebra. Then there is a spectral sequence converging to* $\mathrm{THH}_*(A)$ *with*

$$E^\infty_{p,q} \cong \mathrm{HH}_p^k(A) \otimes_k \mathrm{THH}_q(k).$$

(ii) *If A is as above, and the change of ground-rings map induces a retraction of A-modules $\mathrm{HH}_1^{\mathbb{Z}}(A) \to \mathrm{HH}_1^k(A)$, then any A-linear right-inverse $\mathrm{HH}_1^k(A) \to \mathrm{HH}_1^{\mathbb{Z}}(A)$ induces an A-algebra isomorphism*

$$\mathrm{THH}_*(A) \cong \mathrm{HH}_*^k(A) \otimes_k \mathrm{THH}_*(k).$$

In this paper, all rings are commutative and unital. The base ring will usually be denoted k, and Hochschild homology, $\otimes$, and Tor will always be taken relative to k if no subscript or superscript is given. We often suppress the superscript for λ_* when no confusion is likely to result. Isomorphisms which are both obvious and canonical are often indicated as equalities.

1. H-SMOOTHNESS

Throughout this section, A always denotes a k-algebra.

DEFINITION 1.1: *We say A is* **H-smooth** *over k if $\mathrm{HH}_*(A)$ is flat over k and the Hochschild-Konstant-Rosenberg map λ_n is an isomorphism for all n.*

Note that if k is a $\mathbb{Q}$-algebra the maps λ_n are automatically injective ([9] 1.3.16). Note also that $\mathrm{HH}_0(A) = A$ must be flat over k in order for A to be H-smooth over k.

PROPOSITION 1.2: *Any direct limit of H-smooth k-algebras is H-smooth.*

Proof: Direct limits commute with exterior powers and Hochschild homology. □

PROPOSITION 1.3:

(i) H-smoothness respects arbitrary base change.

(ii) If k' is a faithfully flat k-algebra and $A \otimes k'$ is H-smooth over k', then A is H-smooth over k.

Proof: The Hochschild complex $C_*(A)$ for an H-smooth algebra A is a chain complex of flat modules with flat homology. By the universal coefficient spectral sequence, homology therefore commutes with base change.

The k'-flatness of $\mathrm{HH}_*^{k'}(A \otimes k')$ and faithful flatness of k' over k imply the k-flatness of $\mathrm{HH}_*^k(A)$. Exterior powers also commute with flat base change, so

$$\ker \lambda_*^{A\otimes k'/k'} = \ker \lambda^{A/k} \otimes k'; \quad \operatorname{coker} \lambda_*^{A\otimes k'/k'} = \operatorname{coker} \lambda^{A/k} \otimes k'.$$

As k' is faithfully flat, the condition $\ker \lambda_*^{A\otimes k'/k'} = \operatorname{coker} \lambda_*^{A\otimes k'/k'} = 0$ implies that $\lambda_*^{A/k}$ is an isomorphism. □

THEOREM 1.4: *If A is finitely presented over k then A is H-smooth if and only if it is smooth.*

Proof: One direction is well known. When k is a field, the smoothness of a finitely generated H-smooth k-algebra is due to Avramov and Vigué-Poirrier [2]. See also [4]. For general k the fibers of an H-smooth algebra A are again H-smooth by Proposition 1.3, and therefore smooth. As A is k-flat, it must be smooth ([6] 17.5.1). □

THEOREM 1.5: *A flat k-algebra A is H-smooth if and only if the André-Quillen homology group $D_n(A/k, M) = 0$ for all A-modules M and $n \neq 0$.*

Proof: By [1] Theorem XX.31, A is H-smooth if and only if $D_n(A/A \otimes A, M) = 0$ for all A-modules M and $n \neq 1$. By [*loc. cit.*] Proposition V.21,

$$D_n(A/A \otimes A, M) \xleftarrow{\sim} D_{n-1}(A/k, M)$$

for all $n > 1$. □

COROLLARY 1.6: *Let A be a k-algebra and B an A-algebra. Then if A is H-smooth over k and B is H-smooth over A, B must be H-smooth over A and the relative cotangent sequence*

$$0 \to \mathrm{HH}_1(A) \otimes_A B \to \mathrm{HH}_1(B) \to \mathrm{HH}_1^A(B) \to 0$$

must be exact.

Proof: For the H-smoothness, recall from [1] Theorem V.1 the Jacobi-Zariski sequence

$$\cdots \to D_n(A/k, M) \to D_n(B/k, M) \to D_n(B/A, M) \to D_{n-1}(A/k, M) \to \\ \cdots \to D_0(B/A, M) \to 0$$

which is exact for any B-module M. If $D_n(A/k, M) = D_n(B/A, M) = 0$ for any B-module M and $n > 0$, the exactness gives us that $D_n(B/k, M) = 0$ for any B-module M and $n > 0$.

For the exactness of the relative cotangent sequence, we use the fact ([1] Proposition VI.3) that if Y is an X-algebra,

$$D_0(Y/X, Y) = \mathrm{HH}_1^X(Y).$$

We apply this to all the pairs. Note that since B is flat over A, $D_0(A/k, B) \cong D_0(A/k, A) \otimes_A B$ by [1] Lemma III.20; also $\mathrm{HH}_1(A; B) \cong \mathrm{HH}_1(A) \otimes_A B$. The rightmost end of the Jacobi-Zariski sequence is exactly the desired sequence. □

THEOREM 1.7: *If A is an excellent ring, $\mathfrak{m}$ is a maximal ideal in A, and $\widehat{A}_{\mathfrak{m}}$ is the completion of A at $\mathfrak{m}$, then $\widehat{A}_{\mathfrak{m}}$ is H-smooth over A.*

Proof: As Hochschild homology commutes with localization, $\mathrm{HH}_*^A(A_{\mathfrak{m}})$ is $A_{\mathfrak{m}}$ in degree 0 and zero in higher dimensions, so $A_{\mathfrak{m}}$ is H-smooth over A. By Corollary 31 in the Supplement of [1] and Theorem 1.5 above, $\widehat{A}_{\mathfrak{m}}$ is H-smooth over $A_{\mathfrak{m}}$. Finally, we apply Corollary 1.6. □

2. VALUATION RINGS

Throughout this section, V will always denote a valuation ring with fraction field K, A a dense subring of V with fraction field F, and M an arbitrary V-module. We assume throughout that $(A \cap V^*)^{-1}A = F \cap V$. This will be the case whenever $A = F \cap V$, *i.e.* when A is a valuation subring of V. The cases $\mathbb{Z} \subset \mathbb{Z}_p$ and $E[t] \subset E[\![t]\!]$ also satisfy our hypotheses.

Note that we do not assume that the valuation on V is discrete.

LEMMA 2.1: *If S is the multiplicative system $A \cap V^*$ of elements in A which are invertible in V then $\mathrm{HH}_*^{S^{-1}A}(V, M) = \mathrm{HH}_*^A(V, M)$.*

Proof: If N is any $S^{-1}A$-module,

$$N \otimes_{S^{-1}A} V = N \otimes_{S^{-1}A} S^{-1}V = N \otimes_{S^{-1}A} S^{-1}A \otimes_A V = N \otimes_A V.$$

Thus the Hochschild homology of V with coefficients in M does not depend on whether the ground ring is A or $S^{-1}A$. □

THEOREM 2.2: *For all $i \geq 1$, $\mathrm{HH}_i^A(V, M)$ is a K-vector space.*

Proof: By Lemma 2.1, we may assume without loss of generality that $F \cap V = A$. Let $\bar{V} = V/A$, regarded as A-module. If $a \in A$ is non-zero, multiplication by a induces an isomorphism $\bar{V} \tilde{\to} \bar{V}$. It is injective because $av \in A$ implies $v \in A$, and surjective because we can write any $v \in V$ as $v = w + b$, where $b \in A$ and the valuation of w is greater than that of a. It follows that every non-zero element of A acts invertibly on the truncated normalized Hochschild complex

$$\cdots \to M \otimes_A \bar{V} \otimes_A \bar{V} \to M \otimes_A \bar{V} \to 0$$

obtained from the usual normalized Hochschild complex by omitting the final term M. Therefore every such element acts invertibly on the Hochschild homology groups $\mathrm{HH}_i^A(V, M)$ for $i \geq 1$. As $FV = K$, $\mathrm{HH}_i^A(V, M)$ is a K-vector space. □

Corollary 2.3: *For all* $i \geq 1$, $\mathrm{HH}_i^A(V) = \mathrm{HH}_i^A(K)$.

Proof: As Hochschild homology commutes with localization in the base, for $i \geq 1$,

$$\mathrm{HH}_i^F(K) = \mathrm{HH}_i^A(V) \otimes_A F = \mathrm{HH}_i^A(V).$$

For every F-vector space W, $W \otimes_A K = W \otimes_F K$. Therefore, $\mathrm{HH}_i^A(K) = \mathrm{HH}_i^F(K)$ for all i. □

Lemma 2.4: *If V is complete and H-smooth over A and k is a subring of A such that $\mathrm{HH}_1^k(A)$ is a projective A-module and A is flat over k, then there is a noncanonical isomorphism of V-modules*

$$\mathrm{HH}_1(V) \cong \mathrm{HH}_1(A) \otimes_A V \oplus \mathrm{HH}_1^A(V).$$

Proof: By Corollary 1.6, there is short exact sequence

$$0 \to \mathrm{HH}_1(A) \otimes_A V \to \mathrm{HH}_1(V) \to \mathrm{HH}_1^A(V) \to 0.$$

As $\mathrm{HH}_1(A)$ is projective over A, $\mathrm{HH}_1(A) \otimes_A V$ is projective and hence free over V, while $\mathrm{HH}_1^A(V)$ is a K-vector space. It therefore suffices to prove that $\mathrm{Ext}_V(K, V) = 0$.

The short exact sequence $0 \to V \to K \to K/V \to 0$ gives the isomorphism

$$\mathrm{Ext}_V(K, V) = \mathrm{Hom}_V(K, K/V)/\mathrm{Hom}_V(K, K).$$

If V is ordered by increasing valuation,

$$\mathrm{Hom}(K, K/V) = \mathrm{Hom}(\varinjlim_v v^{-1}V, K/V) = \varprojlim_v K/vV = K = \mathrm{Hom}_V(K, K)$$

since V (and therefore K) is complete. □

Proposition 2.5: *If k is a field and E a finite separable algebraic extension, there are non-canonical isomorphisms*

$$\mathrm{HH}_n^k(E[\![t]\!]) \cong \begin{cases} E[\![t]\!] \oplus W & \text{if } n = 1, \\ \bigwedge^{n-1} W \oplus \bigwedge^n W & \text{if } n \geq 2, \end{cases}$$

where W, the first Hochschild homology of $E((t))$ over $E(t)$, is an infinite-dimensional vector space over $E((t))$.

Proof: As E is an étale k-algebra,

$$HH_n^k(E[\![t]\!]) = \mathrm{HH}_n^k(k[\![t]\!] \otimes_k E) = \mathrm{HH}_n^k(k[\![t]\!]) \otimes_k E.$$

So without loss of generality, we may assume $k = E$. The case $n = 1$ is immediate from Lemma 2.4. As $E[t]$ is smooth over E, $E[\![t]\!]$ is H-smooth over E by Theorem 1.7 and Corollary 1.6, so

$$\begin{aligned} HH_n^E(E[\![t]\!]) &= \bigwedge^n (E[\![t]\!] \oplus W) = \bigoplus_{i+j=n} \bigwedge^i E[\![t]\!] \otimes \bigwedge^j W \\ &= E[\![t]\!] \otimes \bigwedge^{n-1} W \oplus \bigwedge^n W = \bigwedge^{n-1} W \oplus \bigwedge^n W \end{aligned}$$

□

3. TOPOLOGICAL HOCHSCHILD HOMOLOGY OF H-SMOOTH ALGEBRAS

It turns out that the topological Hochschild homology of an H-smooth k-algebra A splits as a product of the topological Hochschild homology of k with the Hochschild homology of A over k. This is quite easy to prove if we are interested only in the linear structure of $\mathrm{THH}_*(A)$; to show that there is in fact an algebra isomorphism we will need the following lemma, which is obvious for degree reasons if $\mathrm{THH}_2(A)$ has no 2-torsion, but is actually true in general.

Lemma 3.1: *Let A be a commutative ring. Then for every $x \in \mathrm{THH}_1(A)$, $x^2 = 0$.*

Proof: If R is any commutative ring and x represents a class in $\mathrm{HH}_1(R)$, we can use the formula for the shuffle product to check directly that $x * x = 0$.

We want to prove a similar result for topological Hochschild homology, but in general we do not have a good map $\mathrm{HH}_*^{\mathbb{Z}}(A) \to \mathrm{THH}_*(A)$. We therefore introduce $\mathbb{Z}[A]$, the free monoid-ring on the elements of A.

By ([7] 6.1), on the spectrum level

$$\mathrm{THH}(\mathbb{Z}[A]) \overset{\sim}{\leftarrow} \mathrm{THH}(\mathbb{Z}) \wedge |N^{\mathrm{cy}}(A)|_+.$$

The obvious map from the right hand side to the left hand side preserves the multiplicative structure, which exists on both sides by A's commutativity. Thus we can use the inclusion of the 0-skeleton $H\mathbb{Z} \overset{i}{\hookrightarrow} \mathrm{THH}(\mathbb{Z})$ (which is multiplicative since the product on THH sends the 0-skeleton smash itself to itself) to get a map

$$\begin{aligned} \mathrm{HH}_*^{\mathbb{Z}}(\mathbb{Z}[A]) = \tilde{H}_*(|N^{\mathrm{cy}}(A)|_+; \mathbb{Z}) &= \pi_*(H\mathbb{Z} \wedge |N^{\mathrm{cy}}(A)|_+) \\ &\overset{i_*}{\to} \pi_*(\mathrm{THH}(\mathbb{Z}) \wedge |N^{\mathrm{cy}}(A)|_+) = \mathrm{THH}_*(\mathbb{Z}[A]). \end{aligned}$$

Note that the inclusion i is 2-connected ([3]), so

$$\mathrm{HH}_1^{\mathbb{Z}}(\mathbb{Z}[A]) \overset{i_1}{\to} \mathrm{THH}_1(\mathbb{Z}[A])$$

is an isomorphism.

Consider now the diagram

$$(3.1.1) \qquad \begin{array}{ccccc} \mathrm{HH}_1^{\mathbb{Z}}(\mathbb{Z}[A]) & \overset{i_1}{\to} & \mathrm{THH}_1(\mathbb{Z}[A]) & \overset{\ell_1}{\to} & \mathrm{HH}_1^{\mathbb{Z}}(\mathbb{Z}[A]) \\ & & \downarrow{\scriptstyle \mathrm{THH}_1(\mathrm{ev})} & & \downarrow{\scriptstyle \mathrm{HH}_1^{\mathbb{Z}}(\mathrm{ev})} \\ & & \mathrm{THH}_1(A) & \overset{\ell_1}{\to} & \mathrm{HH}_1^{\mathbb{Z}}(A), \end{array}$$

where $\mathrm{ev} : \mathbb{Z}[A] \to A$ is the evaluation map, and ℓ_* is the linearization map. Linearization is well-known to be an isomorphism in degree 1, so all the horizontal arrows in (3.1.1) are isomorphisms.

The map $\mathrm{HH}_1(\mathrm{ev}) : \mathrm{HH}_1^{\mathbb{Z}}(\mathbb{Z}[A]) \to \mathrm{HH}_1^{\mathbb{Z}}(A)$ is a surjection, since any cycle $\sum_{i=1}^n a_i \otimes b_i$ is hit by the cycle $\sum_{i=1}^n [a_i] \otimes [b_i]$ (the point here is that in the Hochschild homology complex, $(d_0 - d_1) : R \otimes R \to R$ vanishes for any commutative ring R). We deduce that $\mathrm{THH}_1(\mathrm{ev})$ is surjective, and so the map

$$\mathrm{THH}_*(\mathrm{ev}) \circ i_* : \mathrm{HH}_*^{\mathbb{Z}}(\mathbb{Z}[A]) \to \mathrm{THH}_*(A)$$

is surjective in dimension 1. But this map is multiplicative, so the square of any odd-dimensional element in its image has to be zero. □

The referee pointed out an alternative formulation of the proof, by comparison with Shukla homology. The spectral sequence of Pirashvili and Waldhausen from Remark 4.2 in [12],

$$E^2_{p,q} = \mathrm{Shukla}_p(A, \mathrm{THH}_q(\mathbb{Z}; A)) \Rightarrow \mathrm{THH}_{p+q}(A), \tag{3.1.2}$$

and Bökstedt's calculation of $\mathrm{THH}_*(\mathbb{Z})$ (which is $\mathbb{Z}$ for $* = 0$ and 0 for $* = 1, 2$; see [3]) show us that for $* = 0, 1, 2$,

$$\mathrm{THH}_*(A) \cong \mathrm{Shukla}_*(A). \tag{3.1.3}$$

Another version of this spectral sequence can be deduced from Theorem 6.2.8 of Brun [5]. Let R be a commutative simplicial ring which is free as an abelian group and weakly equivalent to A. Then R defines an FSP which is weakly equivalent to that defined by A, and thus by Brun's Proposition 4.2.5,

$$\mathrm{THH}_*(A) \cong \mathrm{THH}_*(R; A) \cong \pi_*(\mathrm{HH}(\mathbb{Z} \circ R; A)).$$

On the other hand, by [*loc. cit.*] Proposition 4.2.6,

$$\mathrm{Shukla}_*(A) \cong \pi_*(\mathrm{HH}(R; A)).$$

Theorem 6.2.8 of [5] says that the edge homomorphism in the spectral sequence (3.1.2), which induces the isomorphism (3.1.3) in dimensions $0, 1, 2$, is obtained by the FSP evaluation map $\mathbb{Z} \circ R \to R$, and thus is multiplicative by the functoriality of the product on Hochschild homology of a commutative FSP.

On the Shukla complex, as on the Hochschild complex, the (graded) shuffle product is graded-commutative. But whatever the commutative ring A, its Shukla complex is a complex of free abelian groups and so has no 2-torsion, so the square of any representative of a class in $\mathrm{Shukla}_1(A)$ is zero in the Shukla complex. So in homology, all squares of elements in $\mathrm{Shukla}_1(A)$ vanish in $\mathrm{Shukla}_2(A)$, and the corresponding statement in terms of topological Hochschild homology is Lemma 3.1 above.

Theorem 3.2: **(i)** *Let A be an H-smooth k-algebra. Then there is a spectral sequence converging to* $\mathrm{THH}_*(A)$ *with*

$$E^\infty_{p,q} \cong \mathrm{HH}^k_p(A) \otimes_k \mathrm{THH}_q(k).$$

(ii) *If A is as above, and the change of ground-rings map induces a retraction of A-modules $\mathrm{HH}^{\mathbb{Z}}_1(A) \to \mathrm{HH}^k_1(A)$, then any A-linear right-inverse $\mathrm{HH}^k_1(A) \to \mathrm{HH}^{\mathbb{Z}}_1(A)$ induces an A-algebra isomorphism*

$$\mathrm{THH}_*(A) \cong \mathrm{HH}^k_*(A) \otimes_k \mathrm{THH}_*(k).$$

Note that the map $\mathrm{HH}^{\mathbb{Z}}_1(A) \to \mathrm{HH}^k_1(A)$ is a retraction of A-modules in many cases. It is an isomorphism when $\mathrm{HH}^{\mathbb{Z}}_1(k) = 0$, by the relative cotangent sequence in Corollary 1.6. It is a retraction when A is smooth over k, since smoothness makes $\mathrm{HH}^k_1(A)$ a projective A-module; it is also a retraction when k is an algebra over $\mathbb{F}_p$ and $A = k \otimes A'$ for some $\mathbb{F}_p$-algebra A', since in that case $\mathrm{HH}^{\mathbb{Z}}_*(A) = \mathrm{HH}^{\mathbb{Z}}_*(A') \otimes_{\mathbb{F}_p} \mathrm{HH}^{\mathbb{Z}}_*(k)$ and $\mathrm{HH}^k_*(A) = \mathrm{HH}^{\mathbb{Z}}_*(A') \otimes_{\mathbb{F}_p} k$.

Using a simplified linear version of the argument in the proof below, one can show that if A is H-smooth over k, whenever the map $\mathrm{HH}^{\mathbb{Z}}_1(A) \to \mathrm{HH}^k_1(A)$ is a retraction of A-modules,

$$\mathrm{HH}^{\mathbb{Z}}_*(A) \cong \mathrm{HH}^k_*(A) \otimes \mathrm{HH}^{\mathbb{Z}}_*(k)$$

as A-algebras.

Proof of the theorem: Since A is H-smooth over k, it is in particular flat over k so we may use the multiplicative spectral sequence of A-algebras

$$\mathrm{HH}^k_p(A; \mathrm{THH}_q(k) \otimes A) \Rightarrow \mathrm{THH}_{p+q}(A) \tag{3.2.1}$$

from [8] Corollary 3.3 (see also [11] and [5]). Since A is flat over k, the Hochschild homology complex $C^k_*(A)$ is a flat k-complex, quasi-isomorphic to $\mathrm{HH}^k_*(A)$ which is itself flat over k by the H-smoothness. So

$$\begin{aligned}\mathrm{HH}^k_*(A; \mathrm{THH}_*(k) \otimes A) &= \mathcal{H}_*(C^k_*(A) \otimes \mathrm{THH}_*(k)) \\ &= \mathrm{Tor}_*(\mathrm{HH}^k_*(A), \mathrm{THH}_*(k)) = \mathrm{HH}^k_*(A) \otimes \mathrm{THH}_*(k),\end{aligned}$$

and the spectral sequence (3.2.1) becomes simply

$$\mathrm{HH}^k_p(A)\otimes \mathrm{THH}_q(k) \Rightarrow \mathrm{THH}_{p+q}(A). \tag{3.2.2}$$

The E^2-term (3.2.2) is generated multiplicatively by $\mathrm{THH}_*(k)$ in the 0^{th} column and by $\mathrm{HH}^k_1(A)$ sitting in $E^2_{1,0}$; differentials have nowhere to go on these elements, since it is a first-quadrant spectral sequence. Thus the spectral sequence collapses at E^2, and part (i) is established.

To show part (ii), we need to show that the extensions are all trivial, and that the multiplicative structure of $\mathrm{THH}_*(A)$ agrees with the multiplicative structure of the $E^2 = E^\infty$ term (3.2.2).

Observe that the unit map $k \to A$ induces a map

$$\mathrm{THH}_*(k) \to \mathrm{THH}_*(A), \tag{3.2.3}$$

which in terms of the spectral sequence (3.2.2) is just the inclusion

$$\mathrm{THH}_*(k) \xrightarrow{1\otimes \mathrm{id}} 1\otimes \mathrm{THH}_*(k)$$

in the 0'th column.

Since the linearization map

$$\ell_1 : \mathrm{THH}_1(A) \to \mathrm{HH}^{\mathbb{Z}}_1(A)$$

is an isomorphism of A-modules, we can take its inverse,

$$\tilde{s}_1 : \mathrm{HH}^{\mathbb{Z}}_1(A) \to \mathrm{THH}_1(A).$$

As $\mathrm{HH}^k_1(A)$ is a direct summand in $\mathrm{HH}^{\mathbb{Z}}_1(A)$, we may let

$$s_1 : \mathrm{HH}^k_1(A) \to \mathrm{THH}_1(A)$$

denote the restriction of $\tilde{s}_1$ to $\mathrm{HH}^k_1(A)$.

Now $\mathrm{HH}^k_*(A) = \bigwedge^* \mathrm{HH}^k_1(A)$, so we extend s_1 to

$$s_* : \mathrm{HH}^k_*(A) \to \mathrm{THH}_*(A) \tag{3.2.4}$$

multiplicatively. The fact that the squares of elements in $\mathrm{HH}^k_1(A)$ are zero (and consequently also the product $\mathrm{HH}^k_1(A)\otimes \mathrm{HH}^k_1(A) \to \mathrm{HH}^k_2(A)$ is anti-symmetric) will not be a problem by Lemma 3.1.

Note that in terms of the spectral sequence (3.2.2), s_1 maps $\mathrm{HH}_1(A)$ isomorphically onto $E^2_{1,0}$ (one can see this by comparing with the linear version of the spectral sequence, $\mathrm{HH}^k_*(A) \otimes \mathrm{HH}^{\mathbb{Z}}_*(k) \Rightarrow \mathrm{HH}^{\mathbb{Z}}_*(A)$).

Combining (3.2.3) and (3.2.4), we get a map

$$\mathrm{HH}^k_*(A) \otimes \mathrm{THH}_*(k) \xrightarrow{\alpha} \mathrm{THH}_*(A). \tag{3.2.5}$$

If we filter $\mathrm{HH}^k_*(A) \otimes \mathrm{THH}_*(k)$ by the degree of the first factor, the discussion above shows that α is a filtered map. The algebra $\mathrm{HH}^k_*(A) \otimes \mathrm{THH}_*(k)$ is its own associated graded; the tensor product of the maps in (3.2.3) and (3.2.4) maps isomorphically to the E^∞-term (3.2.2) which is the associated graded of $\mathrm{THH}_*(A)$. So by the five-lemma, the map of (3.2.5) is an isomorphism. □

Example 3.3: Let $\mathbb{F}$ be a finite field of characteristic p. Then we have

$$\mathrm{THH}_*(\mathbb{F}[\![t]\!]) \cong \mathrm{HH}_*(\mathbb{F}[\![t]\!]) \otimes \mathrm{THH}_*(\mathbb{F}_p),$$

where $\mathrm{THH}_*(\mathbb{F}_p) \cong \mathbb{F}_p[u_2]$ by [3], and

$$\mathrm{HH}^{\mathbb{F}_p}_n(\mathbb{F}[\![t]\!]) \cong \begin{cases} \mathbb{F}[\![t]\!] \oplus \mathrm{HH}^{\mathbb{F}[t]}_1(\mathbb{F}[\![t]\!]) & \text{if } n = 1, \\ \bigwedge^{n-1} \mathrm{HH}^{\mathbb{F}[t]}_1(\mathbb{F}[\![t]\!]) \oplus \bigwedge^{n} \mathrm{HH}^{\mathbb{F}[t]}_1(\mathbb{F}[\![t]\!]) & \text{if } n \geq 2, \end{cases}$$

by Proposition 2.5. Moreover we know that $\mathrm{HH}^{\mathbb{F}[t]}_1(\mathbb{F}[\![t]\!]) \cong \mathrm{HH}^{\mathbb{F}(t)}_1(\mathbb{F}((t)))$ from the proof of Corollary 2.3, so it is an $\mathbb{F}((t))$-vector space. By [10] Th 86(ii) its dimension is the p-degree of $\mathbb{F}((t))$ over $\mathbb{F}(t)$; as $\mathbb{F}((t))$ has the cardinality of the continuum and $\mathbb{F}(t)$ is countable, this p-degree is the cardinality of the continuum.

Example 3.4: By Theorem 1.7,

$$\mathrm{THH}_*(\mathbb{Z}_p) \cong \bigwedge\nolimits^* \mathrm{HH}_1(\mathbb{Z}_p) \otimes \mathrm{THH}_*(\mathbb{Z}).$$

By [3],

$$\mathrm{THH}_n(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}/k\mathbb{Z} & \text{if } n = 2k-1, \\ 0 & \text{if } n = 2k > 0. \end{cases}$$

By Theorem 2.2, $\mathrm{HH}_1(\mathbb{Z}_p)$ is a $\mathbb{Q}_p$-vector space; its dimension is equal to the transcendence degree of $\mathbb{Q}_p$ over $\mathbb{Q}$ ([10] Th 87(ii)), *i.e.* to the cardinality of the continuum. Thus,

$$\mathrm{THH}_n(\mathbb{Z}_p) \cong \begin{cases} \mathbb{Z}_p & \text{if } n = 0, \\ \mathbb{Z}_p/k\mathbb{Z}_p \oplus \bigwedge^{2k-1} \mathrm{HH}_1(\mathbb{Z}_p) & \text{if } n = 2k-1, \\ \bigwedge^{2k} \mathrm{HH}_1(\mathbb{Z}_p) & \text{if } n = 2k > 0. \end{cases}$$

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