This appendix gives, for all $n \in N$, an affirmative answer to the question of whether there exists a symmetric matrix A in $M_{n\times n}(\mathbf{Z}_2)$ with A^nequal to $2U$, U invertible. I begin by constructing, for each n, an $n \times n$ symmetric matrix with entries in $\mathbf{F}_2$, nilpotent of rank of $n-1$. For n fixed, define $e_{i,j}$ for $1 \leq i, j \leq n$ to be the matrix with $(i,j)^{th}$ entry 1 and all other entries 0. For i or j out of range, let $e_{i,j} = 0$. Let $N = \sum_{i=1}^{n} e_{i,i+1}$ denote the standard nilpotent matrix of rank $n-1$. Over $\mathbf{F}_2$,

$$(I+N)^{-1} = I + N + N^2 + \cdots + N^{n-1}.$$

Define the permuation $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ as follows:

$$\sigma(k) = \begin{cases} 2k & \text{if } k \leq \frac{n}{2} \\ 2(n-k)+1 & \text{otherwise.} \end{cases}$$

Let P_σ denote the corresponding permutation matrix, and

$$M = (I+N)^{-1} P_\sigma^{-1} N P_\sigma (I+N) = \sum_{1 \leq i \leq j \leq n} e_{i,j} \sum_{i=1}^{n-1} e_{\sigma(i),\sigma(i+1)} \sum_{i=1}^{n} (e_{i,i} + e_{i,i+1}).$$

Evidently M is nilpotent of rank $n-1$. I claim that it is symmetric. The cases n even and n odd must be considered separately. Suppose $n = 2m$. Then

$$\sum_{i=1}^{n-1} e_{\sigma(i),\sigma(i+1)} = \sum_{i=1}^{m-1} e_{2i,2i+2} + \sum_{i=1}^{m-1} e_{2i+1,2i-1} + e_{2m,2m-1}.$$

As $e_{i,j} e_{k,\ell} = \delta_{j,k} e_{i,\ell}$,

$$M = \sum_{j=1}^{m-1} \sum_{i=1}^{2j} e_{i,2j+2} + \sum_{j=1}^{m-2} \sum_{i=1}^{2j} e_{i,2j+3} + \sum_{j=1}^{m-1} \sum_{i=1}^{2j+1} e_{i,2j-1} + \sum_{j=1}^{m-1} \sum_{i=1}^{2j+1} e_{i,2j} + \sum_{i=1}^{2m} e_{i,2m-1} + \sum_{i=1}^{2m} e_{i,2m}.$$

Combining the third and fifth terms and likewise the fourth and sixth terms, we get

$$\sum_{j=1}^{m-1} \sum_{i=1}^{2j} e_{i,2j+2} + \sum_{j=1}^{m-2} \sum_{i=1}^{2j} e_{i,2j+3} + \sum_{j=1}^{m} \sum_{i=1}^{2j+1} e_{i,2j-1} + \sum_{j=1}^{m} \sum_{i=1}^{2j+1} e_{i,2j},$$

or (harmlessly) extending the ranges of summation,

$$\begin{aligned} &\sum_{j=0}^{m} \sum_{i=1}^{2j} e_{i,2j+2} + \sum_{j=0}^{m} \sum_{i=1}^{2j} e_{i,2j+3} + \sum_{j=1}^{m} \sum_{i=1}^{2j+1} e_{i,2j-1} + \sum_{j=1}^{m} \sum_{i=1}^{2j+1} e_{i,2j} \\ &= \sum_{k \text{ even}} \left(\sum_{i=1}^{k-2} e_{i,k} + \sum_{i=1}^{k+1} e_{i,k} \right) + \sum_{k \text{ odd}} \left(\sum_{i=1}^{k-3} e_{i,k} + \sum_{i=1}^{k+2} e_{i,k} \right) \\ &= \sum_{k \text{ even}} \sum_{i=k-1}^{k+1} e_{i,k} + \sum_{k \text{ odd}} \sum_{i=k-2}^{k+2} e_{i,k}. \end{aligned}$$

The right hand side is evidently symmetric; the matrix looks like

$$M_{2m} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix}$$

If $n = 2m + 1$,

$$M = \sum_{j=1}^{m} \sum_{i=1}^{2j} e_{i,2j+2} + \sum_{j=1}^{m} \sum_{i=1}^{2j} e_{i,2j+3} + \sum_{j=1}^{m} \sum_{i=1}^{2j+1} e_{i,2j-1} + \sum_{j=1}^{m} \sum_{i=1}^{2j+1} e_{i,2j} + \sum_{i=1}^{2m} e_{i,2m+1}.$$

Combining the third and fifth terms, we get

$$\begin{aligned} M = \sum_{j=1}^{m} \sum_{i=1}^{2j} e_{i,2j+2} + \sum_{j=1}^{m} \sum_{i=1}^{2j} e_{i,2j+3} + \sum_{j=1}^{m+1} \sum_{i=1}^{2j+1} e_{i,2j-1} + \sum_{j=1}^{m} \sum_{i=1}^{2j+1} e_{i,2j} - e_{m+1,m+1} \\ = e_{m+1,m+1} + \sum_{k \text{ even}} \left(\sum_{i=1}^{k-2} e_{i,k} + \sum_{i=1}^{k+1} e_{i,k} \right) + \sum_{k \text{ odd}} \left(\sum_{i=1}^{k-3} e_{i,k} + \sum_{i=1}^{k+2} e_{i,k} \right) \\ = e_{m+1,m+1} + \sum_{k \text{ even}} \sum_{i=k-1}^{k+1} e_{i,k} + \sum_{k \text{ odd}} \sum_{i=k-2}^{k+2} e_{i,k}. \end{aligned}$$

The right hand side is again symmetric. It looks like

$$M_{2m+1} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 \end{pmatrix}$$

To finish the proof, it remains only to show that the matrices constructed in this way can be lifted to $\mathbf{Z}_2$ in such a way that the determininant is in $2\mathbf{Z}_2^*$. The Hensel's lemma argument described above works as longas at least one of the diagonal cofactors $A_{i,i}$ is non-zero. It is easy to prove by induction on n that the cofactor $A_{1,1} \neq 0$. All entries of

M_2, and hence all cofactors, are 1. If $n = 2m > 2$, subtract row $2m$ from row $2m-1$,then row $2m-2$ from row $2m$, and then add row $2m-1$ to row $2m$. If $n = 2m+1$, subtract row $2m$ from row $2m+1$, add row $2m+1$ to row $2m$, then add row $2m+1$ to row $2m-1$. In each case, the row operations leavethe first row untouched and replace M_n by the block diagonal matrix $\begin{pmatrix} M_{n-1} & 0 \\ 0 & (-1)^n \end{pmatrix}$. In fact, not only does this show that the lower left $n \times n$ submatrix of M_n is invertible over $\mathbf{Z}/2\mathbf{Z}$, it even proves invertibility over $\mathbf{Z}$. As the first two rows of M_n are the same, the matrix $X = 2e_{1,1} + M_n$ is a symmetric $n \times n$ matrix such that $\frac{X^n}{2} \in GL_n(\mathbf{Z})$.