

Let E be an elliptic curve defined over a number field K . For each prime ℓ , define

$$T_\ell = \varprojlim_n E_{\ell^n}, \quad V_\ell = T_\ell \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$$

where E_{ℓ^n} denotes the group of ℓ^n -torsion points of E . The Galois group $G_K = \text{Gal}(\bar{K}/K)$ acts continuously on T_ℓ and therefore on V_ℓ . In a series of works ([5], [6], [8], [9]), Serre investigated the image, Γ_ℓ , of G_K in $GL(V_\ell)$. If E has complex multiplication over \mathbf{C} , Γ_ℓ is contained in a Cartan subgroup of $GL(V_\ell)$ (resp. the normalizer of a Cartan) if K contains (resp. does not contain) the endomorphism ring of E . If E does not admit complex multiplication, Γ_ℓ is Zariski-dense in $GL(V_\ell)$ and for all $\ell \gg 0$, $\Gamma_\ell = GL(V_\ell)$. In [11], an analogous result is proved for compatible systems of 2-dimensional Galois representations arising from elliptic modular forms.

Fix a finite set S of primes of K . For the purposes of this paper, a *compatible system of Galois representations, unramified outside S* will be a system of continuous representations

$$\rho_\ell : G_K \rightarrow GL_n(\mathbf{Q}_\ell),$$

one for each rational prime $\ell \notin \prod_{\varphi \in S} \|\varphi\|$, such that

- 1) Each ρ_ℓ is unramified outside $S_\ell = S \cup \{\varphi : \varphi | \ell\}$.
- 2) If $\varphi \notin S_\ell$, the characteristic polynomial of the Frobenius element $\rho_\ell(\text{Frob}_\varphi)$ has coefficients in \mathbf{Q} .
- 3) This characteristic polynomial does not depend on ℓ .

Note that our notion of compatible system is Serre's notion of *strictly compatible system*, [8] I-11. This paper analyzes the dependence of Γ_ℓ on ℓ for two-dimensional representations, using the general philosophy developed jointly with R. Pink in [3] and [4]. By restricting to the two-dimensional case, we can avoid the group-theoretic difficulties which arise in stating and proving analogous statements in higher dimension, and the final results are somewhat stronger than in the general case.

Our main result is the following:

Theorem. *If ρ_ℓ is a compatible system of semisimple 2-dimensional representations, one of the following statements must be true:*

- 1) *The projective representations $\bar{\rho}_\ell : G_K \rightarrow PGL_2(\mathbf{Q}_\ell)$ all factor through a fixed finite quotient $\Gamma \cong A_4$ or $\Gamma \cong S_4$ of G_K .*
- 2) *There exists an extension L/K with $[L : K] \leq 2$ such that the restriction of ρ_ℓ to G_L forms a compatible system of abelian ℓ -adic representations.*
- 3) *For a set of primes ℓ of Dirichlet density 1, there exists a lattice $\Lambda \cong \mathbf{Z}_\ell^2 \subset \mathbf{Q}_\ell^2$ and a quadratic extension L/K , such that $\rho_\ell(G_L)$ is isomorphic to $SL(\Lambda)$.*

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- 4) For all ℓ in a set of primes of Dirichlet density 1, Γ_ℓ is Zariski-dense in $GL_2(\mathbf{Q}_\ell)$, and $\Gamma_\ell/Z(\Gamma_\ell)$ is either $PSL(\Lambda)$ or $PGL(\Lambda)$.

Note that case (1) is trivial, while case (2) is well understood ([8]). Note also that in case (4), Γ_ℓ need not be isomorphic to $GL_2(\mathbf{Z}_\ell)$ for a set of primes of density 1. Consider, for instance, the system obtained from the Tate twist of an elliptic curve:

$$H_{\text{ét}}^1(\bar{E}, \mathbf{Z}_\ell(1)).$$

This system has determinant $\mathbf{Z}_\ell(3)$, so

$$\Gamma_\ell \subset \{M \in GL_2(\mathbf{Z}_\ell) \mid \det(M) \in \mathbf{Z}_\ell^{*3}\}.$$

When $\ell \equiv 1 \pmod{3}$, Γ_ℓ is properly contained in the stabilizer of a lattice Λ ; it cannot be equal to the full stabilizer of a different lattice because $GL(\Lambda') \subset GL(\Lambda)$ if and only if Λ' is homothetic to Λ .

Proposition 1. *Every reductive subgroup G of PGL_2 is either a finite group, a Cartan subgroup, the normalizer of a Cartan subgroup, or the full group.*

Proof. The identity component of G has rank ≤ 1 and is therefore either the trivial group, a rank-1 torus (which is automatically maximal), or PGL_2 . As G is contained in the normalizer of G° , in the torus case, $G^\circ \subset G \subset N(G^\circ)$. The proposition follows from the fact that G° is of index 2 in $N(G^\circ)$. \square

Let $\bar{\Gamma}_\ell$ denote the image of Γ_ℓ under the projection $GL_2(\mathbf{Q}_\ell) \rightarrow PGL_2(\mathbf{Q}_\ell)$. Let G_ℓ denote the Zariski closure of Γ_ℓ in $GL_2(\mathbf{Q}_\ell)$, and \bar{G}_ℓ the Zariski closure of $\bar{\Gamma}_\ell$ in PGL_2 , which is also the image of G_ℓ under $GL_2 \rightarrow PGL_2$. If U is an open and closed subset of $PGL_2(\mathbf{Q}_\ell)$ not containing the identity, the set of elements of $PGL_2(\mathbf{Q}_\ell)$ conjugate to an element of U is open and closed and does not contain the identity. Therefore, if $\bar{\Gamma}_{\ell_0}$ is infinite, we can find an infinite sequence U_i of open subsets of $\bar{\Gamma}_{\ell_0}$, such that no element of U_i is conjugate in $PGL_2(\mathbf{Q}_{\ell_0})$ to any element of U_j , $i \neq j$. By the Chebotarev density theorem, there exists a sequence \wp_i such that $\bar{\rho}_{\ell_0}(\wp_i) \in U_i$. The compatibility of the representations ρ_ℓ implies that for all ℓ , $\bar{\rho}_\ell(\wp_i)$ are all distinct, and $\bar{\Gamma}_\ell$ is infinite for all ℓ .

Proposition 2. *If $\bar{\Gamma}_{\ell_0}$ is finite for some ℓ_0 , then case (1) or case (2) of the main theorem holds.*

Proof. We have seen that all $\bar{\Gamma}_\ell$ must be finite. A semisimple element of $GL_2(\mathbf{Q}_\ell)$ maps to the identity in $PGL_2(\mathbf{Q}_\ell)$ if and only if both eigenvalues are equal and are of the form ± 1 . For Frobenius elements this criterion is independent of ℓ . By the Chebotarev density theorem, the $\bar{\rho}_\ell$ must share a common kernel G_L , an open subgroup of G_K of finite index. The quotient $\Gamma = G_K/G_L$ is a subgroup of $PGL_2(\mathbf{Q}_\ell)$ for all $\ell \notin S$. The kernel of $PGL_2(\mathbf{Q}_\ell) \rightarrow PGL_2(\mathbf{F}_\ell)$ is a pro- ℓ -group, so if $\ell \notin S$ is larger than $|\Gamma|$, we can conclude Γ is a subgroup of $PGL_2(\mathbf{F}_\ell)$. By [2] Ch. XI, Th. 2.3, Γ must be cyclic, dihedral, or isomorphic to one of A_4 , S_4 , or A_5 . Since there are arbitrarily large primes congruent to $\pm 2 \pmod{5}$, by [2] Ch. XI, App., Th. 1, A_5 is ruled out. The cyclic and dihedral cases are subsumed in case (2) of the main theorem; the remaining cases constitute case (1). \square

We assume henceforth that $|\bar{\Gamma}_\ell| = \infty$ for all ℓ . The ℓ -adic Lie algebra associated to Γ_ℓ is of the form $\mathfrak{s} + \mathfrak{c}$, where \mathfrak{s} is the semisimple part of the Lie algebra of $G_\ell(\mathbf{Q}_\ell)$ ([7] Prop. 2). It follows that if \bar{G}_ℓ is PGL_2 , then $\bar{\Gamma}_\ell$ is an open subgroup of $PGL_2(\mathbf{Q}_\ell)$. The adjoint homomorphism $GL_2 \rightarrow SL_3$ factors through PGL_2 , and we use Φ indifferently for the homomorphisms $PGL_2 \rightarrow SL_3$ and $GL_2 \rightarrow SL_3$.

Proposition 3. *Let E_λ be any quadratic extension of \mathbf{Q}_ℓ . Every open subgroup Γ of $PGL_2(\mathbf{Q}_\ell)$ contains an open subset X such that for all $x \in X$, the characteristic polynomial of $\Phi(x)$ has simple roots and generates E_λ , and $\text{tr}(\Phi(x)) \neq -1$.*

Proof. By Krasner's lemma, the splitting field of a polynomial over \mathbf{Q}_ℓ with simple roots is a locally constant function of the coefficients. It suffices, therefore, to find one $x \in \Gamma$ with the desired property. Since the eigenvalues of any $\Phi(x)$ are of the form 1, α , and α^{-1} , the roots will automatically be simple if the splitting field is unequal to \mathbf{Q}_ℓ . If E_λ is a quadratic extension of \mathbf{Q}_ℓ ,

$$PGL_2(\mathbf{Q}_\ell) \cong (\text{Aut}_{\mathbf{Q}_\ell} E_\lambda) / \mathbf{Q}_\ell^*,$$

so $PGL_2(\mathbf{Q}_\ell)$ contains a torus T isomorphic to $E_\lambda^* / \mathbf{Q}_\ell^*$. As $\text{tr} \circ \Phi$ is not identically -1 on T , the regular elements of $x \in T$ with $\text{tr}(\Phi(x)) \neq -1$ are dense in T , so every neighborhood of 1 in $PGL_2(\mathbf{Q}_\ell)$ contains such an x . Any such element will do. \square

Proposition 4. *If for some ℓ_0 , \bar{G}_{ℓ_0} is the normalizer of a Cartan subgroup H , then case (2) holds in the main theorem, with L a quadratic extension of K .*

Proof. Let $\Gamma_{\ell_0}^+$ denote the open index-2 subgroup of Γ_{ℓ_0} whose image $\bar{\Gamma}_{\ell_0}$ lies in the Cartan subgroup H . Then $\rho_{\ell_0}^{-1}(\Gamma_{\ell_0}^+)$ is an open subgroup of G_K of index 2. Let L be the fixed field. Then for all $\text{Frob}(\wp) \in G_K \setminus G_L$, $\rho_{\ell_0}(\text{Frob}(\wp))$ has trace 0. Since the characteristic polynomial of $\rho_\ell(\text{Frob}(\wp))$ doesn't depend on ℓ , it always has trace 0 when \wp is a prime ideal inert in L . By continuity, $\Gamma_\ell^- = \rho_\ell(G_K \setminus G_L)$ is contained in the variety of matrices of trace 0. The Zariski closure of its image in PGL_2 is a coset of the Zariski closure of $\bar{\Gamma}_\ell^+$ in PGL_2 . The closure of $\bar{\Gamma}_\ell^+$ is either the full group \bar{G}_ℓ or an index-2 subgroup thereof. By Prop. 1, \bar{G}_ℓ is a Cartan subgroup, PGL_2 , or the normalizer of a Cartan. In the first two cases, there are no algebraic subgroups of index two, so $\bar{\Gamma}_\ell^+$ is Zariski-dense in \bar{G}_ℓ . It follows that $\bar{\Gamma}_\ell^-$ is Zariski-dense as well. This is inconsistent with $\bar{\Gamma}_\ell^-$ consisting only of matrices of trace 0. We conclude that \bar{G}_ℓ is always the normalizer of a Cartan. \square

Proposition 5. *If H/\mathbf{Q}_ℓ is a torus of GL_2 , there exists $\delta \in \mathbf{Q}_\ell$ such that for all $h \in H(\mathbf{Q}_\ell)$,*

$$\text{disc}(\text{char}(h)) \in \delta \mathbf{Q}_\ell^{*2} \cup \{0\},$$

where $\text{char}(h)$ denotes the characteristic polynomial of h .

Proof. Let $H' = HZ$, where Z is the group of diagonal matrices in GL_2 . Then H' is a torus. Its rank can be no greater than 2, the reductive rank of GL_2 . Therefore, the rational character group, $X^*(H) \otimes_{\mathbf{Z}} \mathbf{Q}$ is a vector space of dimension ≤ 2 with $\text{Gal}(\bar{\mathbf{Q}}_\ell/\mathbf{Q}_\ell)$ action. We have

$$X^*(H) \otimes_{\mathbf{Z}} \mathbf{Q} = (X^*(Z) \otimes_{\mathbf{Z}} \mathbf{Q}) \oplus M,$$

where the Galois action on $X^*(Z)$ is trivial, and $\dim(M) \leq 1$. As a subgroup of $\text{Aut}_{\mathbf{Q}}(M)$, the image of the Galois group is compact and discrete, hence finite, hence of order ≤ 2 . Therefore, H splits either over \mathbf{Q}_ℓ or over $\mathbf{Q}_\ell(\sqrt{\delta})$ for some $\delta \in \mathbf{Q}_\ell$. The eigenvalues of $h \in H(\mathbf{Q}_\ell)$ are respectively two elements of \mathbf{Q}_ℓ or a conjugate pair of elements of $\mathbf{Q}_\ell(\sqrt{\delta})$. In either case, the discriminant of $\text{char}(h)$ is δ times a perfect square. \square

Corollary. *Under the hypotheses of Proposition 5, $\Phi(h)$ has eigenvalues $\{1, \alpha, \alpha^{-1}\}$, where $\alpha \in \mathbf{Q}_\ell$ if $\delta \in \mathbf{Q}_\ell^{*2}$, and otherwise α lies in the norm kernel*

$$\ker(\text{N}_{\mathbf{Q}_\ell(\sqrt{\delta})/\mathbf{Q}_\ell}).$$

Proof. We know that α is the ratio of the two eigenvalues of h . If H is \mathbf{Q}_ℓ -split, the ratio lies in \mathbf{Q}_ℓ and $\delta \in \mathbf{Q}_\ell^{*2}$. If not, α and α^{-1} are conjugate elements of $\mathbf{Q}_\ell(\sqrt{\delta})$. In particular, if $\alpha \in \mathbf{Q}_\ell$ it must belong to $\{\pm 1\}$. \square

Proposition 6. *Let $\Gamma \subset \text{PGL}_2(\mathbf{Q}_\ell)$ be a compact subgroup with elements γ_1 and γ_2 such that modulo ℓ , $\Phi(\gamma_1)$ has 3 distinct eigenvalues in \mathbf{F}_ℓ , while $\Phi(\gamma_2)$ has at least one eigenvalue not in \mathbf{F}_ℓ . Suppose also that $\ell > 3$, the (mod ℓ) reduction of either $\Phi(\gamma_1)$ or $\Phi(\gamma_2)$ has order > 5 in $SL_3(\mathbf{F}_\ell)$, and $\text{tr}(\Phi(\gamma_2))$ is not congruent to -1 (mod ℓ). Then Γ contains $PSL_2(\mathbf{Z}_\ell)$.*

Proof. Let $\Gamma' \supset \Gamma$ be a maximal compact subgroup of $\text{PGL}_2(\mathbf{Q}_\ell)$. We claim that, up to conjugation, there are two possibilities for Γ' : $\text{PGL}_2(\mathbf{Z}_\ell)$ and

$$\begin{aligned} \mathcal{M} = & \left\{ \bar{M}|M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \det(M) \in \mathbf{Z}_\ell^*, \ell|c \right\} \\ & \cup \left\{ \bar{M}|M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \det(M) \in \ell\mathbf{Z}_\ell^*, \ell|a, \ell|c, \ell|d \right\}. \end{aligned}$$

This is a consequence of the general theory of semisimple groups over local fields [1], but it is not difficult to prove directly: Let $GL_2^+(\mathbf{Q}_\ell)$ denote the subgroup of $GL_2(\mathbf{Q}_\ell)$ of elements with unit determinants, and let

$$\pi : GL_2^+ \rightarrow \text{PGL}_2^+ = \{\bar{M}|M \in GL_2(\mathbf{Q}_\ell), \det(M) \in \ell^{2\mathbf{Z}}\mathbf{Z}_\ell^*\}$$

denote the projection map. As π has compact kernel, the inverse image of

$$\Gamma^+ = \Gamma' \cap \text{PGL}_2^+,$$

is a compact subgroup of GL_2 and therefore stabilizes a rank-2 \mathbf{Z}_ℓ -lattice Λ . If Γ' stabilizes Λ up to homothety, it is contained in $\text{PGL}(\Lambda)$, so by maximality it equals $\text{PGL}_2(\mathbf{Z}_\ell)$. Otherwise, there exists $\gamma' \in \Gamma' \setminus \Gamma^+$ such that $\gamma'\Lambda \not\sim \Lambda$. By the normality of index-2 subgroups, Γ^+ stabilizes $\gamma\Lambda$ as well as Λ . Now, the homothety classes of \mathbf{Z}_ℓ -lattices in \mathbf{Q}_ℓ^2 are the vertices of the *Bruhat-Tits tree*, where edges connect two lattices when one is of index ℓ in the other. The group $\text{PGL}_2(\mathbf{Q}_\ell)$ acts on this tree. If $\gamma^+ \in \Gamma^+$ fixes two vertices, it must also fix every vertex on the unique path connecting them. On the other hand, γ'

exchanges Λ and $\gamma'\Lambda$ (as $\gamma'^2 \in \Gamma^+$), so it reverses the order of the intermediate vertices. If the number of edges between Λ and $\gamma'\Lambda$ is even, there is a central vertex which is fixed by Γ^+ and γ' , hence by Γ . If the number is odd, there is a pair of adjacent vertices which are fixed by Γ^+ and exchanged by γ' . Choosing e_i so that these vertices correspond to the lattices $\mathbf{Z}_\ell e_1 + \mathbf{Z}_\ell e_2$ and $\mathbf{Z}_\ell e_1 + \ell\mathbf{Z}_\ell e_2$, we obtain the group \mathcal{M} defined above.

The image of the identity component of \mathcal{M} under Φ is contained in the Iwahori subgroup of $SL_3(\mathbf{Z}_\ell)$, that is, the subgroup of elements which are upper-triangular (mod ℓ). The image of the non-identity component is a subset of the 3×3 matrices with trace congruent to -1 (mod ℓ). Therefore, $\gamma_2 \notin \mathcal{M}$. It follows that $\Gamma \subset PGL_2(\mathbf{Z}_\ell)$. Let $\tilde{\Gamma}$ denote the pre-image of Γ in the projection $GL_2(\mathbf{Z}_\ell) \rightarrow PGL_2(\mathbf{Z}_\ell)$ and $\tilde{\gamma}_i$ an element of $\tilde{\Gamma}$ lying over γ_i . By [10] Lemma 2, every proper subgroup of $GL_2(\mathbf{F}_\ell)$ which fails to contain $SL_2(\mathbf{F}_\ell)$ lies either in a Borel subgroup, in the normalizer of a Cartan subgroup, or in the pre-image in $GL_2(\mathbf{F}_\ell)$ of one of $A_4, S_4, A_5 \subset PGL_2(\mathbf{F}_\ell)$. As every element of A_4, S_4 , and A_5 has order ≤ 5 , these possibilities are ruled out for the (mod ℓ) reduction of $\tilde{\Gamma}$. If the trace of $\tilde{\gamma}_i$ were divisible by ℓ , the ratio of the eigenvalues would be congruent to -1 (mod ℓ), so two of the eigenvalues of $\Phi(\gamma_i)$ would be congruent to -1 , contrary to hypothesis. This means that if $\tilde{\gamma}_i$ lies in the normalizer of a (mod ℓ) Cartan, it lies in the Cartan itself. By hypothesis, $\tilde{\gamma}_1$ can lie only in a split Cartan, while $\tilde{\gamma}_2$ can lie only in a non-split Cartan. Moreover, $\tilde{\gamma}_2$ is not contained in a Borel. Therefore, the reduction (mod ℓ) of $SL_2(\mathbf{Z}_\ell) \cap \tilde{\Gamma}$ is $SL_2(\mathbf{F}_\ell)$. By [10] Lemma 1, $\tilde{\Gamma} \supset SL_2(\mathbf{Z}_\ell)$, so $\Gamma \supset PSL_2(\mathbf{Z}_\ell)$. \square

We can now prove the main theorem. By Prop. 4, we may assume that for all ℓ , \bar{G}_ℓ is either a Cartan subgroup of PGL_2 or the full group. Suppose that for some ℓ_0 , it is all of PGL_2 . Then there exist primes \wp_1 and \wp_2 such that the eigenvalues of $\Phi(\rho_{\ell_0}(Frob(\wp_i)))$ generate distinct quadratic extensions $\mathbf{Q}_{\ell_0}(\sqrt{\delta_i})$. Let α_i denote an eigenvalue which generates $\mathbf{Q}_{\ell_0}(\sqrt{\delta_i})$. In particular, $\alpha_i \neq \pm 1$. For $S, T \subset \mathbf{Q}^{ab}$, the maximal abelian extension of \mathbf{Q} , let

$$L(S, T) = \{\ell \text{ prime} \mid S \subset \mathbf{Q}_\ell \text{ and } T \subset \bar{\mathbf{Q}}_\ell \setminus \mathbf{Q}_\ell\}.$$

Note that this does not depend on the embedding $\bar{\mathbf{Q}} \subset \bar{\mathbf{Q}}_\ell$ because the elements of S and T lie in abelian extensions of \mathbf{Q} . If $\ell \in L(\{\alpha_1\}, \{\alpha_2\})$, then by the corollary to Prop. 5, $\bar{G}_\ell = PGL_2$. Indeed, no single Cartan subgroup of PGL_2/\mathbf{Q}_ℓ can contain elements whose Φ -images have eigenvalues in $\mathbf{Q}_\ell \setminus \{\pm 1\}$ and $\bar{\mathbf{Q}}_\ell \setminus \mathbf{Q}_\ell$. By the Chebotarev density theorem, $L(S, T)$ is infinite whenever S and T are finite. Therefore, if $\bar{G}_\ell = PGL_2$ holds for one ℓ , it holds for infinitely many.

Now assume that $\bar{G}_{\ell_i} = PGL_2$ for primes ℓ_1, ℓ_2, \dots . Applying Prop. 3, we obtain for each i a prime \wp_i such that $\text{tr}(\Phi(\wp_i)) \neq -1$ and an eigenvalue β_i of $\Phi(Frob(\wp_i))$ which generates a ramified extension of \mathbf{Q}_{ℓ_i} . The total extension

$$\Omega = \mathbf{Q}(\beta_1, \beta_2, \dots)/\mathbf{Q}$$

is infinite because it is ramified at every prime ℓ_i . We extract an infinite subsequence $\gamma_1 = \beta_{f(1)}, \gamma_2 = \beta_{f(2)}, \dots$ such that $f(1) < f(2) < \dots$,

$$[\mathbf{Q}(\gamma_1, \dots, \gamma_{n+1}) : \mathbf{Q}(\gamma_1, \dots, \gamma_n)] = 2,$$

and $\gamma_i \notin \mathbf{Q}(\zeta_{60})$. We apply Prop. 6 to the set of $Frob(\gamma_i)$, $i = 1, 2, \dots, n$. Every rational prime ℓ not dividing

$$N_n = N_{K/\mathbf{Q}} \left(\prod_{i=1}^n \wp_{f(i)} \right) \prod_{i=1}^n disc(\gamma_i^{60} - 1) \prod_{i=1}^n N_{\mathbf{Q}(\gamma_i)/\mathbf{Q}}(\text{tr}(\Phi(Frob(\wp_{f(i)}))) + 1),$$

and not contained in

$$\ell \notin L(\{\gamma_1, \dots, \gamma_n\}, \{\}) \cup L(\{\}, \{\gamma_1, \dots, \gamma_n\})$$

satisfies $\bar{G}_\ell = PGL_2$. By the Chebotarev density theorem (in this case, actually, the Dirichlet density theorem is enough), the density of $L(\{\gamma_1, \dots, \gamma_n\}, \{\})$ is 2^{-n} and the density of $L(\{\}, \{\gamma_1, \dots, \gamma_n\})$ is 2^{-n} . As $N_n \neq 0$, the set of primes dividing N_n is finite and therefore of density zero. For every prime ℓ not dividing N_n , the eigenvalues of $Frob(\wp_{f(i)})$ are distinct and of order $> 5 \pmod{\ell}$. By Prop. 6, the set of primes such that $\bar{G}_\ell \supset PSL_2(\mathbf{Z}_\ell)$ is at least $1 - 2^{1-n}$. Taking the limit as $n \rightarrow \infty$, we conclude immediately that $\bar{G}_\ell = PSL_2(\mathbf{Z}_\ell)$ or $\bar{G}_\ell = PGL_2(\mathbf{Z}_\ell)$ for a set of primes ℓ of density 1.

Now, $\det(\rho_\ell)$ forms a compatible system of 1-dimensional Galois representations. Its image is finite for some ℓ if and the whole system factors through a homomorphism $G_K \rightarrow \mu_n$, where μ_n denotes the group of n^{th} roots of unity. For this to happen, $\mu_n \subset \mathbf{Q}_\ell$ for all ℓ , which means $n = 1$ or $n = 2$. Letting L be the fixed field of the kernel of $G_K \rightarrow \mu_n$, we see that condition (3) of the theorem is satisfied. If $\det(\rho_\ell)$ has infinite image, the image is Zariski dense in $GL_1(\mathbf{Q}_\ell)$, so if $\bar{G}_\ell \supset PSL_2(\mathbf{Z}_\ell)$, $G_\ell = GL_2$, $Z(\Gamma_\ell) = \Gamma_\ell \cap Z(GL_2(\mathbf{Q}_\ell))$, and condition (4) is satisfied. \square

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