Representation Zeta Functions

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Representation Zeta Functions

Let G be a group. We define

$$\zeta_G(s) = \sum_V (\dim V)^{-s}.$$

The sum is taken over all irreducible finite-dimensional complex representations of G.

What Kinds of Group?

1. Compact Lie groups, e.g. $\mathrm{SU}(n)$.
2. Compact p-adic groups, e.g. $\mathrm{SL}(n, \mathbf{Z}_p)$.
3. Finitely generated groups, e.g. $\mathrm{SL}(n, \mathbf{Z})$.

Note: A finite-dimensional complex representation of $\mathrm{SL}(n, \mathbf{Z}_p)$ is continuous if and only if it has finite image.

Motivating example: If $G = \mathrm{SU}(2)$, then

$$\zeta_G(s) = \zeta(s).$$

Basic Question

Is $\zeta_G(s)$ really a zeta function?

Properties of a Good Zeta Function

0. Convergence in a half-plane: $\sum_n n^{-s}$ converges for $\Re(s) > 1$.
1. Euler product:

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$

2. Interesting special values:

$$\zeta(2n) = \frac{-B_{2n}}{2(2n)!}(2\pi i)^{2n}.$$

3. Analytic continuation: For $\zeta(s)$, this can be achieved in two steps, first to $\Re(s) > 0$ and then to the whole plane, via the:
4. Functional equation:

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}\zeta(1-s).$$

5. Riemann hypothesis?

Convergence in a Half-Plane

Convergence is known in the following cases:

1. For G a compact semisimple Lie group, e.g. $\mathrm{SU}(n)$.
2. For G a compact open subgroup of a semisimple group over a p-adic field, e.g. $\mathrm{SL}(n, \mathbf{Z}_p)$.
3. For $G = \Gamma$ an arithmetic group satisfying the congruence subgroup property, e.g. $\mathrm{SL}(n, \mathbf{Z})$, for $n \geq 3$.

We write $\rho(G)$ for the *abscissa of convergence.*

Euler Products

There is an Euler product of sorts for arithmetic groups satisfying the congruence subgroup property. For example, for $n \geq 3$,

$$\zeta_{\mathrm{SL}(n,\mathbf{Z})}(s) = \zeta_{\mathrm{SU}(n)}(s) \prod_p \zeta_{\mathrm{SL}(n,\mathbf{Z}_p)}(s).$$

Note that $\mathrm{SL}(n, \mathbf{Z}_p)$ is not quite a pro-p group (inverse limit of finite groups of p-power order). For pro-p groups, all irreducible representations are of p-power degree, so the zeta function is a power series in p^{-s}.

Special Values

Theorem (Witten, 1991). *If G is a compact semisimple Lie group associated to a root system Φ and n is a positive integer, then*

$$\zeta_G(2n) \in \pi^{n|\Phi|}\mathbf{Q}.$$

The value also has an interpretation as the volume of a natural geometric object.

Analytic Continuation

Theorem (Jaikin-Zaparain, 2006). *For G a compact open subgroup of a semisimple group over a p-adic field (p odd), $\zeta_G(s)$ extends to an entire meromorphic function.*

Question: Is the same true for semisimple compact Lie groups?

Gassmann-Sunada Phenomenon

Theorem (Gassmann, 1926). *There exist number fields K_1 and K_2 such that*

1. $\zeta_{K_1}(s) = \zeta_{K_2}(s)$
2. *The fields K_1 and K_2 are not isomorphic*
3. *The minimal Galois extensions of K_1 and K_2 over $\mathbf{Q}$ are the same.*

Theorem (Sunada, 1985). *There exist compact Riemannian manifolds X_1 and X_2 such that*

1. $\zeta_{X_1}(s) = \zeta_{X_2}(s)$
2. *The manifolds X_1 and X_2 are not isometric*
3. *The universal covers $\tilde{X}_1$ and $\tilde{X}_2$ are the same.*

Gassmann-Sunada Construction

Data for the construction:

1. Subgroups H_1 and H_2 of a finite group G.
2. A bijection $\phi : H_1 \to H_2$.
3. A geometric object on which G acts faithfully.

Conditions:

4. For all $h \in H_1$, $\phi(h)$ is conjugate to h in G.
5. H_1 and H_2 are not conjugate in G.

Example: $G = S_6$ and

$$H_1 = \{e, (34)(56), (56)(12), (12)(34)\}$$

$$H_2 = \{e, (12)(34), (13)(24), (14)(23)\}$$

An Analogue for Lie Groups

Theorem (Larsen, 2004). *There exist compact semisimple Lie group* G_1 *and* G_2 *such that*

1. $\zeta_{G_1}(s) = \zeta_{G_2}(s)$
2. *The groups* G_1 *and* G_2 *are not isomorphic*
3. *The universal covers* $\tilde{G}_1$ *and* $\tilde{G}_2$ *are the same.*

Moreover, if G_1 and G_2 satisfy (1) and (2), they also satisfy (3).

Using Outer Automorphisms

Difficulty: *Finite normal subgroups of compact connected Lie groups lie in the center.*

Modified Gassmann-Sunada conditions:

4'. For all $h \in H_1$, $\phi(h)$ is equivalent to h under $\mathrm{Aut}(G)$.

5'. H_1 and H_2 are not equivalent under $\mathrm{Aut}(G)$.

Lubotzky's Program

Let $R_n(G)$ denote the number of irreducible representations of degree $\leq n$.

Basic Fact. The abscissa of convergence of $\zeta_G(s)$ is

$$\limsup_{n\to\infty} \frac{\log R_n(G)}{\log n}.$$

Roughly: $R_n(G)$ grows like $n^{\rho(G)}$.

1. Find the abscissa of convergence for linear groups.
2. Relate it to other measures of growth in group theory (e.g. subgroup growth).

Induction and Restriction

Let $H < G$ be a subgroup of index m. Then

$$R_n(G) \leq mR_n(H) \leq m^2 R_{mn}(G).$$

Proposition (Lubotzky-Martin, 2004). *The abscissas of convergence for commensurable groups are the same.*

Some Open Questions

1. What is $\rho(\mathrm{SL}(n, \mathbf{Z}))$? This is open even for $n = 3$.

2. What is $\rho(\mathrm{SL}(n, \mathbf{Z}_p))$? This is open even for $n = 3$.

3. What is the limiting behavior of either of the above quantities as $n \to \infty$?

Compact Lie Groups

Theorem. *Let G be a compact simple Lie group with root system Φ. Then*

$$\rho(G) = \frac{2\,\mathrm{rank}(\Phi)}{|\Phi|}.$$

Example. $\rho(\mathrm{SU}(n)) = \frac{2}{n}$.

Idea of proof. For example,

$$\zeta_{\mathrm{SU}(3)} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(ij(i+j)/2)^{-s}.$$

The set $\{i, j \geq 1 \mid ij(i+j) < n^3\}$ has order $O(n^2)$.

The P-adic Analogue

Theorem. *Let G be a simple algebraic group with (absolute) root system Φ over a local field K and U an open subgroup of $G(K)$. Then*

$$\rho(U) \geq \frac{2\,\mathrm{rank}(\Phi)}{|\Phi|}.$$

Example. $\rho(\mathrm{SL}(n, \mathbf{Z}_p)) \geq \frac{2}{n}$.

Idea of the Proof

Every irrep. of $\mathrm{SL}(3, \mathbf{Z}/p^k\mathbf{Z})$ is an irrep. of $\mathrm{SL}(3, \mathbf{Z}_p)$. Diagonal elements

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (ab)^{-1} \end{pmatrix}$$

in $\mathrm{SL}(3, \mathbf{Z}/p^k\mathbf{Z})$ give $\sim p^{2k}$ conjugacy classes, hence at least that many irreducible representations.

The sum of the squares of degrees of all irreps of $\mathrm{SL}(3, \mathbf{Z}/p^k\mathbf{Z})$ is $\sim p^{8k}$. So a typical irrep. has degree $O(p^{3k})$.
Thus,

$$R_{p^{3k}}(\mathrm{SL}(3, \mathbf{Z}/p^k\mathbf{Z})) \gg p^{2k}.$$

What About Non-diagonal Classes?

For semisimple groups over *fields*, "most" classes are semisimple. The number of diagonal classes in $\mathrm{SL}(n, \mathbf{Z}/p^k\mathbf{Z})$ grows like $p^{(n-1)k}$.

Let $G^\natural$ denote the set of conjuacy classes in G.

Proposition. *For fixed n and p*

$$\lim_{k\to\infty} \frac{\log_p |\mathrm{SL}(n, \mathbf{Z}/p^k\mathbf{Z})^\natural|}{k} \geq \frac{n^2 - 4n + 3}{12}.$$

Idea. It is enough to count classes represented by block matrices of the form

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

where A and D are congruent to identity matrices (mod $p^{n/3}$).

Non-semisimple Bounds

An immediate consequence is:

$$\liminf_{n\to\infty} \rho(\mathrm{SL}_n(\mathbf{Z}_p)) \geq \frac{2}{11}.$$

Similar bounds apply for other "classical" families of p-adic groups.

Isotropic vs. Anisotropic

A simple algebraic group G over a field K is *isotropic* if and only if it contains a subgroup isomorphic to $\mathrm{GL}(1)$; otherwise, it is *anisotropic*.

For local fields K, the following are equivalent for simple G:

1. G is anisotropic over K.
2. $G(K)$ is compact.
3. $G(K) = D^*/K^*$, where D is a division algebra with center K.

Local Dichotomy

Theorem (Larsen-Lubotzky). *Let G be a simple algebraic group over a local field K. If $G(K)$ is compact then for any compact open subgroup $U \subset G(K)$,*

$$\rho(U) = \frac{2\,\mathrm{rank}(\Phi)}{|\Phi|}.$$

Theorem (Larsen-Lubotzky). *Let G be a simple algebraic group over a local field K. If $G(K)$ is not compact then for any compact open subgroup $U \subset G(K)$,*

$$\rho(U) \geq \frac{1}{15}.$$

Basic Inequality

If $G \to H$ is a surjective homomorphism,

$$\rho(G) \geq \rho(H).$$

If H is a topological group, it suffices that the image of G is dense.

Example. $\rho(\mathrm{SL}(n, \mathbf{Z})) \geq \rho(\mathrm{SL}(n, \mathbf{Z}_p))$.

Gap Theorem

Theorem (Larsen-Lubotzky). *If* Γ *is an infinite linear group which is finitely generated, then*

$$\rho(\Gamma) \geq \frac{1}{15}.$$

Theorem (Larsen-Lubotzky). *There exists an infinite, finitely generated, residually finite group* Γ *such that*

$$\rho(\Gamma) = 0.$$

Lattices

A *lattice* Γ in a semisimple Lie group G is a discrete subgroup such that G/Γ has finite volume.

A lattice is *irreducible* if its image in every quotient of G is dense. This rules out lattices like $\Gamma_1 \times \Gamma_2 \subset G_1 \times G_2$.

Conjecture. Any two irreducible lattices in the same semisimple Lie group have the same abscissa of convergence.

Computable Examples

Let $K_1, \ldots, K_n$ be finite extensions of $\mathbf{R}$ or of $\mathbf{Q}_p$.

Theorem (Larsen-Lubotzky). *If* Γ *is an irreducible lattice in* $\prod_i \mathrm{SL}_2(K_i)$*, then*

$$\rho(\Gamma) = \begin{cases} \infty & \textit{if } n = 1, \\ 2 & \textit{otherwise.} \end{cases}$$