Math S212 Exam 3 Name: Michael Larsen

For each convergence problem, you should not only indicate whether the sum in question converges or diverges. You should also give a reason, such as "p-test" or "ratio test" and substantiating details if necessary. (For instance, if you say "limit comparison test", you should at least indicate the series you are comparing to and how you know whether that series converges or diverges.)

1. (10 pts) Evaluate the sum $3/2 + 1 + 2/3 + (2/3)^2 + (2/3)^3 + \cdots$ or indicate if it does not converge.

   This is a geometric series with $a = 3/2$ and $r = 2/3$, so it converges with sum $9/2$.

2. (10 pts) Evaluate the sum

   $$(\sin 0 - \sin \pi/2) + (\sin \pi/2 - \sin \pi) + (\sin \pi - \sin 3\pi/2) + (\sin 3\pi/2 - \sin 2\pi) + \cdots$$

   or indicate if it does not converge.

   The nth term has absolute value 1 for all n, so it doesn't converge.

3. (10 pts) Does the series $\frac{10}{1!} - \frac{10^2}{2!} + \frac{10^3}{3!} - \cdots$ converge, and if so, is the convergence conditional or absolute?

   It converges absolutely by the ratio test (since $10/n \to 0$ as $n \to \infty$).

4. (10 pts) Does $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$ converge or diverge?

   It converges by the ratio test (limit $1/2$) or the limit comparison test (comparison to $1/2^n$, which converges as a geometric series with $r = 1/2$.)

5. (10 pts) Does $\sum_{n=2}^{\infty} \frac{\sqrt{n+1}}{n(n-1)}$ converge or diverge?

It converges by limit comparison to $\frac{\sqrt{n}}{n^2} = n^{-3/2}$, which converges by the p-test.

6. (10 pts) Does $\frac{1}{2\ln 2} - \frac{1}{3\ln 3} + \frac{1}{4\ln 4} - \cdots$ converge, and if so, is the convergence conditional or absolute?

   This converges by the alternating series test. It does not converge absolutely, since the integral test gives convergence if and only if $\int_2^\infty \frac{dx}{x\ln x}$ converges; the indefinite integral is $\ln\ln x + C$, and this goes to ∞ with x.

7. (10 pts) Does $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^2$ converge or diverge?

   This converges. You can do a comparison to $n^{-3/2}$ (which converges by the p-test). As $n \to \infty$, $n^{-1/4}\ln n \to 0$ (by L'Hôpital), so for big enough n, $n^{-1/2}\ln^2 n < 1$, or $\ln^2 n < \sqrt{n}$). Or you can use the integral test, substituting $u = \ln x$ and using integration by parts to write

   $$\int \frac{\ln^2 x\, dx}{x^2} = \int u^2 e^{-u}\, du = (-u^2+2u-2)e^{-u} = \frac{\ln^2 x + 2\ln x - 2}{x} + C.$$

   This goes to 0 as x goes to infinity, again by L'Hôpital's rule.

8. (10 pts) Does $\sum_{n=1}^{\infty} \frac{2^n - 1}{n^n}$ converge or diverge?

   This converges by comparison to $\frac{2^n}{n^n}$, which converges by the root test (since $2/n \to 0$ as $n \to \infty$).

9. (10 pts) Does $\sum_{n=1}^{\infty} \frac{2+\sin n}{n}$ converge or diverge?

   This diverges by the comparison test since the nth term is at least $1/n$, which diverges by the p-test.

10. (10 pts) The sequence $a_n = \frac{n}{n+1}$ converges to 1. That means, for all $\epsilon > 0$, there exists N such that for all $n \geq N$, $|a_n - 1| < \epsilon$. Find a formula for N in terms of ϵ which shows this is true.

    Take $N = 1/\epsilon$. Then

    $$|a_n - 1| = 1/(n+1) < 1/n \leq 1/N = \epsilon.$$

BONUS (10 pts) Does $\sum_{n=1}^{\infty}(1-\cos(1/n))$ converge or diverge?

Use L'Hôpital's rule (twice) to check that $\frac{1-\cos(1/n)}{n^{-2}} \to \frac{1}{2}$ as $n \to \infty$. Then use limit comparison and the p-test to deduce convergence.