Math S212 Exam 2 Name:

1. (12 pts) What estimate does Simpson's rule give for $\int_0^1 x^2\,dx$ if you evaluate only at 0, 1/2, and 1?

$$\frac{1\cdot 0^2 + 4\cdot(1/2)^2 + 1\cdot 1^2}{6} = 1/3.$$

It's right on the nose.

2. (12 pts) Evaluate the integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ or indicate if it diverges and why.

We have

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2} = 2\int_{0}^{\infty} \frac{dx}{1+x^2}$$

and

$$\int_{0}^{\infty} \frac{dx}{1+x^2} = \lim_{t\to\infty}\int_0^t \frac{dx}{1+x^2} = \lim_{t\to\infty} \tan^{-1} t = \pi/2,$$

so the answer is $\pi$.

3. (20 pts) Find the area of the region bounded by the line $y+2x=0$ and the parabola $y=x^2$. Find the volume of the solid obtained by rotating this region about the $x$-axis.

Solving the two equations, we get $x=0$ and $x=-2$, and on $[-2,0]$, $-2x > x^2$. The area is

$$\int_{-2}^{0} -2x - x^2\,dx = -x^2 - x^3/3\Big]_{-2}^{0} = \frac{4}{3}.$$

The solid of revolution has volume

$$\int_{-2}^{0} \pi(2x)^2 - \pi(x^2)^2\,dx = \frac{4\pi x^3}{3} - \frac{\pi x^5}{5}\Big]_{-2}^{0} = \frac{64\pi}{15}.$$

4. (12 pts) Find the volume of the solid of revolution obtained by rotating the figure bounded by the coordinate axes and the curve $y=\cos^{-1} x$ around the $x$-axis.

Equivalently, find the volume of the solid of revolution obtained by rotating the figure bounded by the coordinate axes and the curve $x = \cos^{-1} y$ around the $y$-axis. The curve $x = \cos^{-1} y$ can be rewritten $y = \cos x$. Setting $y = 0$, we get $x = \pi/2$, so $x$ ranges from 0 to $\pi/2$. Using the method of cylindrical shells, this gives

$$\int_0^{\pi/2} 2\pi x \cos x \, dx.$$

Integrating by parts, we get

$$2\pi(x \sin x + \cos x)\Big]_0^{\pi/2} = \pi^2 - 2\pi.$$

5. (12 pts) Show that the mean value theorem for integrals is true for the function $f(x) = x^2$ on the interval $[1, 4]$ by finding the value of the number $c$ giving the mean value.

   The mean value is

$$\frac{1}{4-1} \int_1^4 x^2 \, dx = 7,$$

   so $c = \pm\sqrt{7}$. The mean value theorem is supposed to give a value in the interval, so $c = \sqrt{7}$.

6. (12 pts) If a spring has natural length 10 cm. and it takes 20 J of work to stretch it from 10 cm. to 20 cm., how much work does it take to compress it from 10 cm. to 5 cm.?

   If the spring constant is $k$, then $F = kx$, where $x$ is the length minus natural length. Thus,

$$20 = \int_0^{.1} kx \, dx = .01k/2,$$

   so $k = 4000$ (in units of $J/m^2$). Therefore,

$$\int_0^{.05} kx \, dx = (.05)^2 k/2 = 5J$$

   .

7. (20 pts) Find the arc length of the curve $y = \cosh x = \frac{e^x+e^{-x}}{2}$ from $x = 1$ to $x = 2$. Find the area of the surface of revolution obtained by rotating this arc around the $x$-axis.

As $\frac{dy}{dx} = \sinh x$, we have

$$ds = \sqrt{1 + \sinh^2 x}\, dx = \cosh x\, dx,$$

so the arc length from $x = 1$ to $x = 2$ is

$$\int_1^2 \cosh x\, dx = \sinh x\Big]_1^2 = \sinh 2 - \sinh 1 = \frac{e^2 - e + 1/e - 1/e^2}{2}.$$

For surface area, we integrate $2\pi y\, ds$:

$$2\pi \int_1^2 \cosh^2 x\, dx = 2\pi \int_1^2 \frac{e^{2x} + e^{-2x} + 2}{4}\, dx = \pi \frac{e^4 - e^2 - e^{-2} + e^{-4} + 4}{4}.$$

BONUS (10 pts) Evaluate the integral $\int_0^\infty \frac{dx}{x^2(x-1)^2}$ or indicate if it diverges and why.

The integral is improper because the function is not defined at 0 or 1 and the integral goes to $\infty$. We break up the integral into pieces from 0 to 1/2, from 1/2 to 1, from 1 to 2, and from 2 to $\infty$ so that each piece is improper for only one reason. As $\frac{1}{x^2(x-1)^2} > \frac{1}{x^2}$ for $x < 1$, the divergence of $\int_0^{1/2} \frac{dx}{x^2}$ implies the divergence of $\int_0^\infty \frac{dx}{x^2(x-1)^2}$.