Math S212 Exam 1 Name: Michael Larsen

1. (10 pts) If $f(x) = \frac{\ln x}{2} + 1$, what is $f^{-1}(x)$? What is its domain and range?

   If $y = \frac{\ln x}{2} + 1$, then $y - 1 = \frac{\ln x}{2}$, $2y - 2 = \ln x$, and $x = e^{2y-2}$. Thus $f^{-1}(x) = e^{2x-2}$. The domain is the set of all real numbers. The range is the domain of $f(x)$, i.e., the set of positive numbers.

2. (10 pts) Compute $\lim_{x\to\infty} \frac{(\ln x)^4}{x}$.

   By L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{(\ln x)^4}{x} = \lim_{x\to\infty} \frac{4(\ln x)^3}{x} = \lim_{x\to\infty} \frac{12(\ln x)^2}{x}$$
$$= \lim_{x\to\infty} \frac{24(\ln x)}{x} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

3. (16 pts) Evaluate $\int_0^1 \frac{2x^3}{(x^2+1)^2}\, dx$.

   Write

$$\frac{2x^3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2},$$

   and solve for A, B, C, D: $A = 2$, $C = -2$, $B = D = 0$. Substituting $u = x^2 + 1$,

$$\int_0^1 \frac{2x^3}{(x^2+1)^2}\, dx = \int_0^1 \frac{2x}{x^2+1} - \frac{2x}{(x^2+1)^2}\, dx$$
$$= \int_1^2 \frac{1}{u} - \frac{1}{u^2}\, du = \ln 2 - \frac{1}{2}.$$

4. (16 pts) Evaluate $\int_0^{\pi/2} \cos^4 x\, dx$.

Use trig identities to write

$$\int_0^{\pi/2} \cos^4 x\, dx = \frac{1}{4}\int_0^{\pi/2} (\cos 2x + 1)^2\, dx$$
$$= \frac{1}{8}\int_0^{\pi/2} \cos 4x + 4\cos 2x + 3\, dx,$$

and substitution to evaluate the integral and obtain $\frac{3\pi}{16}$.

5. (16 pts) Evaluate $\int \frac{\sqrt{x^2-4x}}{x-2}\, dx$.

Substitute $y = x-2$ to rewrite the integral as $\int \frac{\sqrt{y^2-2^2}}{y}\, dy$, and then substitute $y = 2\sec u$ to obtain

$$2\int \tan^2 u\, du = 2\int (\sec^2 u - 1)\, du = 2\tan u - 2u + C$$
$$= \sqrt{y^2-2^2} - 2\sec^{-1} y/2 + C = \sqrt{x^2-4x} - 2\cos^{-1}\frac{2}{x-2} + C.$$

6. (16 pts) Evaluate $\int e^{t^{1/3}}\, dt$.

Substitute $x = t^{1/3}$ to obtain $3\int x^2 e^x\, dx$. Integrate by parts twice to obtain

$$3\int x^2 e^x\, dx = 3x^2e^x - 6\int xe^x = (3x^2 - 6x + 6)e^x + C$$
$$(3t^{2/3} - 6t^{1/3} + 6)e^{t^{1/3}} + C.$$

7. (16 pts) Evaluate $\int x \sec^2 x\, dx$.

Integrate by parts to get

$$\int x \sec^2 x\, dx = x \tan x - \int \tan x\, dx = x \tan x - \ln|\sec x| + C.$$

BONUS (10 pts) Evaluate $\int \frac{dx}{3+5\sin x}$.

Substitute $u = \tan(x/2)$ to obtain

$$\int \frac{dx}{3+5\sin x} = \int \frac{2/3\, du}{u^2 + (10/3)u + 1}.$$

Using partial fractions, we can rewrite this integral as

$$\int \frac{1/4\, du}{u+1/3} - \int \frac{1/4\, du}{u+3} = \frac{\ln|\tan(x/2)+1/3| - \ln|\tan(x/2)+3|}{4} + C.$$