Putnam Competition Outtake Solutions

1. Let $n = m^2 + k$. If $k \ll m$, by the binomial theorem,

$$n^{3/2} = m^3 + (3/2)mk + (3/8)m^{-1}k^2 + O(m^{-3}k^3).$$

Choosing m even and sufficiently large, and (for $0 < \alpha < 1$)

$$k = \left\lfloor \sqrt{\frac{8m\alpha}{3}} \right\rfloor,$$

we can make the fractional part of $n^{3/2}$ approximate α to any given accuracy. In particular, we set $\alpha = .19965$.

2. By a suitable choice of coordinates, we may assume that $P_1(t) = P_1$ and $P_2(t) = P_2$ are constant. We define $x_i(t)$ and $y_i(t)$ to be the coordinates of $P_i(t)$ for $3 \leq i \leq 4$. Consider the following conditions:

(a) $x_3(t) + x_4(t) \geq -2$ (b) $x_3(t) + x_4(t) \leq 2$

(c) $y_3(t) \geq 0$ (d) $y_4(t) \leq 0$

These are closed conditions and are satisfied for $t = 0$. If they hold for all t, then all four of the distances specified in the problem remain bounded at $t \to \infty$. Suppose not. Let t_0 be the greatest lower bound of the set of t such that one or more of the above conditions fails. Then t_0 itself satisfies all four conditions but equality holds in one or more. We consider the possibilities.

(a) The point P_2 and the midpoint M of $P_3(t_0)P_4(t_0)$ lie on a common vertical line, and the distance from M to $P_3(t_0)$ (resp. $P_4(t_0)$) is 1. If $y_3(t) + y_4(t) \geq 0$, then by (d), $d(P_2, P_4(t_0)) \leq d(M, P_4(t_0)) = 1$, contrary to assumption. If $y_3(t) + y_4(t) \leq 0$, then by (c), $d(P_2, P_3(t_0)) \leq d(M, P_3(t_0)) = 1$, contrary to assumption.

(c) If $P_3(t_0) = (x_3(t_0), 0)$ is more than one unit away from both $(-1, 0)$ and $(1, 0)$, then $x_3(t_0) < -2$ or $x_3(t_0) > 2$. Either way, the midpoint of the length-2 segment $P_3(t_0)P_4(t_0)$ has an x-coordinate with absolute value > 1, contrary to (a) and (b).

Cases (b) and (d) follow by symmetry from (a) and (c) respectively.

3. The linear transformation $(x, y) \mapsto (x, y\sqrt{2})$ multiplies areas by $\sqrt{2}$ and maps the ellipse into the unit circle. Rotate the coordinate system so that the chord AC is horizontal and B lies above AC. Replacing B by $(0, 1)$ preserves the inequality in triangle areas and increases the area of ABC. Likewise, we replace D either by a point as far below the line BC as possible, given the area constraint. If the x coordinate of BC is t, the area of the resulting quadrilateral is

$$\frac{1}{2}\sqrt{1 - x^2}\,\inf(2, (3/2)(1 - x)).$$

The maximum of this function is

$$\sup\Big(\sup_{x \in [-1/3, 1]} (3/4)(1 - x)\sqrt{1 - x^2}, \sup_{x \in [-1, -1/3]} \sqrt{1 - x^2}\Big).$$

The critical points of $(1-x)^2(1-x^2)$ are at $x=-1/2$ and $x=1$, so

$$\sup_{x\in[-1/3,1]} (3/4)(1-x)\sqrt{1-x^2} = \sqrt{8/9} = \sup_{x\in[-1,-1/3]} \sqrt{1-x^2}.$$

We conclude that the maximum area for the original quadrilateral is 2/3.

4. Let r denote the smallest radius of any circle in S, C_r, the set of centers of all circles in S with radius r. Let $c \in S$ denote a circle whose center is a point x of C_r which is on the boundary of the convex hull of C_r. Let $x_1, \ldots, x_n$ denote the centers of the circles in S which are tangent to c and r_i the radii of those circles. The minimal angle between $x - x_i$ and $x - x_j$ is $\leq 2\pi/n$, so if $n \geq 6$, by the law of cosines,

$$\begin{aligned}(r_i + r_j)^2 &\leq (r + r_i)^2 + (r + r_j)^2 - (r + r_i)(r + r_j) \\ &= r^2 + rr_i + rr_j + r_i^2 + r_j^2 - r_i r_j.\end{aligned}$$

As $r \leq r_i$ and $r \leq r_j$, we must have equality everywhere; in particular, $n = 6$, $r = r_i = r_j$, and the points x_i form a regular hexagon centered at x. This is impossible since the x_i must all belong to C_r, and x is in the interior of the convex hull of the x_i.

We conclude by noting that for $n \leq 5$ there do exist such sets. Take a regular polyhedron inscribed in a sphere. Project the centers of faces and midpoints of edges from the center onto the sphere. The projections of centers are then the centers of circles passing through the projections of midpoints. Now take a stereographic projection onto the plane. That settles the case $3 \leq n \leq 5$. The cases $n = 1$ and $n = 2$ are trivial.

5. Let $n = 4y^4 + 4y^2$.

6. Let $G(s)$ denote the area of the region

$$R(s) = \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ (1-x)^{1/s} + (1-y)^{1/s} \geq 1\}.$$

Thus $4 - F(s) = 4G(s)$. Substituting $u = x/s$, $v = y/s$, we conclude that $s^{-2}G(s)$ is the area of $[0, 1/s] \times [0, 1/s]$ lying below the curve

$$(1 - su)^{1/s} + (1 - sv)^{1/s} = 1,$$

which is the same as the curve

$$\exp(-u - su^2/2 - s^2u^3/3 - \cdots) + \exp(-v - sv^2/2 - s^2v^3/3 - \cdots) = 1.$$

By the monotone convergence theorem,

$$\lim_{s\to\infty} \frac{G(s)}{s^2} = \int_0^\infty -\log(1 - e^{-u})du.$$

Substituting $t = e^{-u}$, we obtain the integral

$$\int_0^1 \frac{-\log(1-t)dt}{t} = \int_0^1 \sum_{n=1}^\infty \frac{t^n}{n}\frac{dt}{t},$$

or using positivity to justify the interchange of sum and integral,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Putting it all together,

$$\lim_{s \to 0} \frac{4 - F(s)}{s^2} = \frac{2\pi^2}{3}.$$

7. Let $F_i(x)$ denote the formal power series whose n^{th} coefficient is the x^{n+i} coefficient of $(1 + x + x^2)^n$. Thus $F_{-i} = F_i$, and

$$F_i - x(F_{i-1} + F_i + F_{i+1}) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases} \qquad (*)$$

Thus, there exist formal Laurent series $\alpha(x), \beta(x), \gamma(x), \delta(x)$ such that for $i \geq 0$,

$$F_i = \gamma(x)\alpha(x)^i + \delta(x)\beta(x)^i,$$

and

$$(y - \alpha(x))(y - \beta(x)) = y^2 + (1 - 1/x)y + 1.$$

Solving,

$$\alpha(x) = \frac{1 - x + \sqrt{1 - 2x - 3x^2}}{2x}, \ \beta(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x},$$

and as $\alpha(x)$ has leading term $\frac{1}{x}$ while $\beta(x)$ (like all the F_i) is a power series, $\gamma(x) = 0$. Setting $i = 0$ in (*),

$$1 = (1 - x - 2x\beta(x))\delta(x),$$

or

$$F_0(x) = \delta(x) = (1 - 2x - 3x^2)^{-1/2}.$$

8. By an easy induction, the number of non-contiguous subsets of a polygonal path of length n is F_n, the n^{th} Fibonacci number. Thus

$$f(n) = F_n + F_{n-2} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)}{2}.$$

It follows that

$$f(4n) + 2 = f(2n)^2.$$

9. We observe that

$$(aB - Ab)^2 + (aA + bB)^2 = (a^2 + b^2)(A^2 + B^2)$$

is a perfect square, so there exist integers x, y, z with either

$$aB - Ab = (2xy)z, aA + bB = (x^2 - y^2)z, \sqrt{(a^2+b^2)(A^2+B^2)} = (x^2+y^2)z$$

or

$$aB - Ab = (x^2 - y^2)z, aA + bB = (2xy)z, \sqrt{(a^2+b^2)(A^2+B^2)} = (x^2+y^2)z.$$

By hypothesis, $aB - Ab \neq 0$, so in the first case $x \neq 0, y \neq 0$ and in the second case $x \neq \pm y$. Thus in the first case,

$$(aB - Ab)^2 = 4x^2y^2z^2 > 2x^2y^2z \geq (x^2+y^2)z = \sqrt{(a^2+b^2)(A^2+B^2)} \geq bB$$

and in the second case,

$$(aB - Ab)^2 = (x^2-y^2)^2z^2 > (x^2+y^2)z = \sqrt{(a^2+b^2)(A^2+B^2)} \geq bB.$$

Taking square roots of both sides,

$$|a/b - A/B| = \frac{aB - Ab}{bB} > \frac{\sqrt{bB}}{bB} = \frac{1}{\sqrt{bB}}.$$

10. Substitute $x = \sqrt{a}, y = \sqrt{b}, z = \sqrt{c}$. We can rewrite the inequality as

$$(x+y+z)(-x+y+z)(x-y+z)(x+y-z) \leq \frac{(x+y+z)^4}{27}.$$

If x, y, z fail to satisfy the triangle inequality, then the inequality is trivial since the left hand side is non-positive and right hand side non-negative. If they do satisfy the triangle inequality, this is the isoperimetric theorem for triangles.

11. Setting $b_n = \sqrt{2a_n - 1}$, we have

$$b_{n+1} = \frac{b_n}{a_n} = \frac{2b_n}{b_n^2+1} = \tanh\left(2\tanh^{-1} b_n\right).$$

Therefore,

$$1 = \lim_{n\to\infty} b_n = \lim_{n\to\infty} b_1 \prod_{i=1}^{n-1} a_i^{-1} = \frac{\sqrt{2a_1 - 1}}{\prod_{n=1}^{\infty} a_i}$$

as long as $b_1 > 0$. In particular, when $a_1 = 2$, the product is $\sqrt{3}$.

12. Otherwise, the (mod 1) reductions of $x_1, x_1+x_2, \ldots, x_1+\cdots+x_n$ all lie in the interval $[1/(n+1), n/(n+1)]$. The result follows immediately from the pigeon-hole principle.

13. Let p be a prime dividing n. Any solution (mod n) is certainly a solution (mod p), so without loss of generality we may assume n is prime. Also, the points $P_i = (x_i, y_1)$ are

pairwise distinct (mod n), since the distances between pairs are prime to n. If $n = 2$, there are only 4 different points in $(\mathbf{Z}/n\mathbf{Z})^2$, and their pairwise distances are not all the same. Therefore, we may assume n is odd. In the field $\mathbf{Z}/n\mathbf{Z}$

$$\begin{aligned}(6P_1 - 2P_2 - 2P_3 - 2P_4) \cdot (P_2 - P_3)& \\ &= 6P_1 \cdot P_2 - 6P_1 \cdot P_3 - 2P_2^2 + 2P_3^2 - 2P_4 \cdot P_2 + 2P_4 \cdot P_3 \\ &= 3(P_1 - P_3)^2 - 3(P_1 - P_2)^2 + (P_4 - P_2)^2 - (P_4 - P_3)^2 = 0.\end{aligned}$$

Likewise, the dot product of $(6P_1 - 2P_2 - 2P_3 - 2P_4)$ with $P_2 - P_4$ is zero in $\mathbf{Z}/n\mathbf{Z}$. Thus either $3P_1 = P_2 + P_3 + P_4$, or $P_2 - P_3$ and $P_2 - P_4$ are collinear. However, $P_2 - P_3$, $P_3 - P_4$, and $P_4 - P_2$ all have the same norm, so if they are collinear, two must coincide. Without loss of generality, we may assume $P_2 - P_3 = P_3 - P_4$. Setting $Q = P_1 - P_3$, $R = P_2 - P_3 = P_3 - P_4$, we have

$$Q^2 = (Q + R)^2 = (Q - R)^2,$$

or, since $p \neq 2$, $Q \cdot R = R^2 = 0$. Since $R \neq 0$, this implies that R and Q are proportional, so $Q^2 = 0$, contrary to hypothesis. We conclude that $3P_1 = P_2 + P_3 + P_4$ and by symmetry that $3P_2 = P_1 + P_3 + P_4$. Subtracting, we deduce that $2(P_1 - P_2) = 0$, so $P_1 = P_2$, a contradiction.

14. By induction on n, we see that if $BA = qAB$,

$$(A + B)^n = \sum_{m=0}^{n} \binom{n}{m}_q A^m B^{n-m},$$

where

$$\binom{n}{m}_q = \frac{\prod_{i=1}^{n} \frac{q^i - 1}{q - 1}}{\prod_{j=1}^{m} \frac{q^j - 1}{q - 1} \prod_{k=1}^{n-m} \frac{q^k - 1}{q - 1}}.$$

Setting $q = e^{2\pi i/n}$ the result is immediate.

15. Let

$$f(x, y) = 1 + \sum_{i=1}^{\infty} Q_i(x) y^i.$$

Differentiating the recursive formula for Q_n, we obtain

$$f(x, y) f_x(x, y) = y.$$

Regarding y as a parameter and solving the above differential equation in x by separation of variables, we obtain

$$f(x, y)^2 = C(y) + 2xy,$$

where C is a constant of integration depending on y. The recursion formula also implies $f(x, 1) = 1 + y$, so $C(y) = 1 + y^2$, and

$$f(x, y) = \sqrt{1 + 2xy + y^2}.$$

Setting $x = 0$, we obtain a power series expansion in y^2, so the coefficients $Q_{2k+1}(0)$ are zero.

16. Equivalently every linear transformation T of a 2-dimensional vector space over the field $\mathbf{Z}/p\mathbf{Z}$ satisfies $T^{p^3} = T$. If T has an eigenvector, then it can be represented as the upper triangular matrix

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

By induction,

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^n = \begin{pmatrix} a^n & b(a^{n-1} + a_{n-2}d + \cdots + d^{n-1}) \\ 0 & d^n \end{pmatrix},$$

so the identity follows from Fermat's little theorem. If T has no eigenvector it is invertible, and for every non-zero vector v, the vectors v and Tv space the vector space. Therefore, $T^a = T^b$ if and only if $T^a v = T^b v$ and $T^{a+1}v = T^{b+1}v$. Moreover, the latter condition implies the former. Thus $T^a v = T^b v$ if and only if $T^a v' = T^b v'$, for any non-zero vector v'. Hence, the length of the orbit $v, Tv, T^2v, \ldots, T^{k-1}v = T^{-1}v$ is independent of v. Since there are $p^2 - 1$ non-zero vectors, k divides $p^2 - 1$, so $T^{p^2-1} = 1$, or $T^{p^2} = T$, from which the desired identity follows immediately.

17. Let

$$S_{m,n} = \{(x, y) \mid x \in [m - 1/2, m + 1/2), y \in [n - 1/2, n + 1/2)\}.$$

Let R denote the region enclosed by C and $\mu(X)$ denote the area of any set $X \in \mathbf{R}^2$. Thus

$$A(C) = \mu(R) = \sum_{m,n \in \mathbf{Z}} \mu(R \cap S_{m,n}) \leq N(C) + \sum_{(m,n) \in \mathbf{Z}^2 \setminus R} \mu(R \cap S_{m,n}).$$

If $(m, n) \notin R$, then $R \cap S_{m,n}$ is a proper subset of $S_{m,n}$. If it is non-empty, then the line segment connecting any point in it with (m, n) meets the curve C at some point in $S_{m,n}$. Therefore, C contains a point in $S_{m,n}$.

Let $k = \lceil L(C)/2\sqrt{2} \rceil$. Choose $x_0, \ldots x_{k-1}, x_k = x_0$ on C such that the distance between x_i and x_{i+1} is $\leq 2\sqrt{2}$. Any point within distance $\sqrt{2}$ of C is within distance 2 of some point x_i. Therefore if $C \cap S_{m,n}$ is non-empty, $S_{m,n}$ lies in a disk of radius 2 around some x_i. Hence

$$4\pi k \geq |\{(m, n) \in \mathbf{Z}^2 \setminus R \mid \mu(R \cap S_{m,n}) > 0\}|,$$

so

$$A(C) \leq N(C) + 4\pi \lceil L(C)/2\sqrt{2} \rceil \leq N(C) + \pi\sqrt{2}L(C) + 4\pi$$

This gives an inequality of the desired kind as long as $L(C)$ is bounded from below. For $L(C) \leq 2\pi$, however, the isoperimetric theorem implies $A(C) \leq L(C)/2$, so the inequality holds again.

18. The vector space of degree-n polynomials in x and y is $(n + 1)$-dimensional. Setting $z = x + iy$, the expressions $z^i \bar{z}^{n-i}$ belong to this vector space, and there are $n + 1$ such.

Regarded as functions on the unit circle, they coincide with the functions $z^n, z^{n-2}, \ldots, z^{-n}$, which are obviously linearly independent since a Laurent polynomial has only finitely many zeroes. The powers $z^i, i \in \mathbf{Z}$ are mutually orthogonal on the unit circle. Thus the vector space of degree-n polynomials orthogonal to all polynomials of lower degree on the unit circle is spanned by z^n and z^{-n}. These are meromorphic functions, hence harmonic. Finally, it is easy to see that a harmonic homogeneous polynomial of degree n in x and y is determined by its x^n and $x^{n-1}y$ coefficients, so the vector space of such polynomials is at most two dimensional, therefore equal to the span of z^n and z^{-n}.

19. If the degree d is large enough and $P(x), P(x-1) \in X_n$, there exists $m < d - n$ such that neither $P(x)$ nor $P(x-1)$ has a term of degree m, $m+1$, ..., or $m+n$. If $Q(x) = P^{(m)}(x)$, then

$$Q(x) = \sum_{i=1}^{n} b_i x^{m_i}, \; m_i \geq n,$$

and $Q^{(k)}(1) = 0$ for $0 \leq k < n$. Therefore,

$$\sum_{i=1}^{n} b_i m_i^k =, \; 0 \leq k < n.$$

As the determinant of the Vandermonde matrix $||m_i^j||$ is non-zero, all the b_i must be zero, so $Q(x)$ is zero, contrary to assumption.

20. Clearly $\sin 1, \cos 1 \in S$. As $\pi < 4$, $\sin 1 > \cos 1$. Given an interval $I_i = [c,d] \subset [0,1]$, either $c > \sin 1$; or $c \leq \sin 1$ and $d < \cos 1$; or $S \cap I_i$ is non-empty. In the first case $I_{i+1} = [\cos^{-1} d, \cos^{-1} c] \subset [0,1]$ and in the second $I_{i+1} = [\sin^{-1} c, \sin^{-1} d] \subset [0,1]$; either way if $S \cap I_{i+1}$ is non-empty, the same is true for $S \cap I_i$. If $[c,d] \cap S$ is empty, setting $I_0 = [c,d]$, there is an infinite sequence $I_0, I_1, I_2, \ldots$ of intervals disjoint from S. As $\sin^{-1}$ and $-\cos^{-1}$ have derivatives > 1 on $(0,1)$, the length of I_{i+1} exceeds the length of I_i. Indeed, the sequence $|I_i|$ is monotonically increasing and bounded above, but it cannot converge. The contradiction proves the theorem.