

## Integration by Parts

The Leibniz rule for differentiation says that if  $f(x) = g(x)h(x)$ , then

$$f'(x) = g'(x)h(x) + g(x)h'(x).$$

By the fundamental theorem of calculus

$$\int g'(x)h(x) + g(x)h'(x) dx = \int f'(x) dx = f(x)$$

(ignoring constants of integration). The indefinite integral (i.e., the antiderivative) of a sum of two functions is the sum of the indefinite integrals, so

$$\int g'(x)h(x) + g(x)h'(x) dx = \int g'(x)h(x) dx + \int g(x)h'(x) dx.$$

Putting it all together,

$$\int g'(x)h(x) dx + \int g(x)h'(x) dx = f(x) = g(x)h(x).$$

Subtracting  $\int g(x)h'(x) dx$  from both sides of the equation, we get the formula for integration by parts:

$$\int g'(x)h(x) dx = g(x)h(x) - \int g(x)h'(x) dx. \quad (*)$$

The reason (\*) is useful is that it gives a way to integrate a product of two functions, in this case the functions  $g'(x)$  and  $h(x)$ . Given a product which is to be integrated by this method, it is important to choose which factor should be called  $g'(x)$  and which one  $h(x)$ . One consideration to bear in mind is that the right hand side is expressed in terms of  $g(x)$ , not  $g'(x)$ , which means we had better choose a  $g'(x)$  whose indefinite integral we know. Another is that it is desirable if the derivative of  $h(x)$  is in some sense a simpler function than  $h(x)$ , or at least no more complicated. For instance,  $h(x) = \ln x$  makes a good choice since  $h'(x) = \frac{1}{x}$  is a more basic function than the natural log. So does  $h(x) = x$ , whose derivative  $h'(x) = 1$  is a constant (and constants are nice because we can pull them through integral signs). A function like  $h(x) = e^x$  isn't too bad; its derivative  $e^x$  is at least no worse than the original function. The point of all this is that at the end of the day we will need to integrate  $g(x)h'(x)$ , and the trick of integration by parts is only worth doing if this integral is easier to work out than the original one.

**Example 1.** Compute the indefinite integral  $\int e^x x dx$ .

There are two factors  $e^x$  and  $x$ . We can easily compute the indefinite integral of either one. Both have fairly simple derivatives, but  $x$  is especially nice because its derivative is a constant. So set  $g'(x) = e^x$  and  $h(x) = x$ . Then  $h'(x) = 1$ , and  $g(x)$  could be any function of the form  $e^x + C$ . We might as well choose the simplest one,  $g(x) = e^x$ . Now (\*) says

$$\int e^x x dx = e^x x - \int e^x dx = e^x x - e^x + C.$$

What would have happened if we had made the opposite choice? Well,  $g'(x) = x$ ,  $h(x) = e^x$  implies  $h'(x) = e^x$ , and  $g(x) = \frac{x^2}{2} + C$ . Making the simplest choice,  $g(x) = \frac{x^2}{2}$ , which means (\*) says

$$\int e^x x \, dx = x e^x - \int \frac{x^2}{2} e^x \, dx.$$

The integral on the right hand side is actually harder to do than the original integral, so we've lost ground by making injudicious choices.

**Example 2.** Compute the indefinite integral  $\int x^2 \sin x \, dx$ .

There are several possible factorizations since  $x^2 \sin x = x \cdot x \cdot \sin x$  is a product of three terms. One reasonable choice is to take  $g'(x) = \sin x$  and  $h(x) = x^2$ . Then  $h'(x) = 2x$ , and we can take  $g(x) = -\cos x$ . Applying (\*),

$$\int x^2 \sin x \, dx = -x^2 \cos x - \int (-\cos x)(2x) \, dx = -x^2 \cos x + 2 \int x \cos x \, dx.$$

There is still an unevaluated integral, but  $x \cos x$  looks like it might be a little easier to integrate than  $x^2 \sin x$ . Proceeding on that assumption, let's try to evaluate  $\int x \cos x \, dx$  by setting  $g'(x) = \cos x$  and  $h(x) = x$ . This means setting  $h'(x) = 1$  (very promising) and  $g(x) = \sin x$ . Plugging into (\*) again,

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C,$$

and going back to the original problem,

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2 \int x \cos x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C.$$

This is complicated enough that it's worth taking a derivative and seeing if we get the original function. Indeed, the derivative of  $-x^2 \cos x + 2x \sin x + 2 \cos x + C$  is

$$(-2x \cos x + x^2 \sin x) + (2 \sin x + 2x \cos x) - 2 \sin x = x^2 \sin x.$$

Moral: sometimes applying integration by parts once isn't enough.

**Example 3.** Compute the indefinite integral  $\int \ln x \, dx$ .

This one doesn't look like an integration by parts problem at all (at first glance). But try writing  $\ln x = g'(x)h(x)$ , where  $g'(x) = 1$  and  $h(x) = \ln x$ . This means  $g(x) = x$  and  $h'(x) = 1/x$ . Applying (\*),

$$\int \ln x \, dx = x \ln x - \int x \frac{1}{x} \, dx = x \ln x - \int dx = x \ln x - x + C.$$

## Problems

1. Evaluate  $\int x \sin x \, dx$ .
2. Evaluate  $\int x \ln x \, dx$ .
3. Evaluate  $\int x \cos 2x \, dx$ .
4. Evaluate  $\int (x^2 + x + 1)e^x \, dx$ .
5. Evaluate  $\int x e^{-x}$ .