Math M120 **Exam 3** April 4, 2003

Name: Michael Larsen

1. (20 pts) Suppose the supply curve for a certain commodity is given by $p = q^2 - q + 1$ and the demand curve by $p = 17 - q$.

a) Find the equilibrium quantity and price.
b) Find the producer surplus.
c) Find the consumer surplus.
d) Find the total gains from trade.

Setting $q^2 - q + 1 = 17 - q$*, we get* $q^2 = 16$*, and solving (since quantity cannot be negative),* $q^* = 4$*. Plugging in to either equation,* $p^* = 13$*. The producer surplus is given by*

$$\int_0^4 13 - (q^2 - q + 1)\, dq = \int_0^4 12 - q + q^2\, dq = 48 + 8 - \frac{64}{3} = \frac{104}{3}.$$

The consumer surplus is given by

$$\int_0^4 (17 - q) - 13\, dq = \int_0^4 4 - q\, dq = 16 - 8 = 8.$$

Gains from trade are computed by adding consumer and producer surplus: $\frac{104}{3} + 8 = \frac{128}{3}$.

2. (10 pts) Assuming a continuous interest rate of 4%, what is the present value of an annuity paying \$10,000 per year, beginning 25 years from today and ending 50 years from today?

The formula for present value gives

$$\int_{25}^{50} 10000e^{-.04t}\, dt = 10000 \int_{25}^{50} e^{-.04t}\, dt.$$

Now either by the method of substitution or the boxed formula on the bottom of p. 280, $F(t) = \frac{e^{-.04t}}{-.04} = -25e^{-.04t}$ *is an anti-derivative of* $e^{-.04t}$*, so*

$$\int_{25}^{50} 10000e^{-.04t} = 10000(F(50) - F(25)) = 250000\left(\frac{1}{e} - \frac{1}{e^2}\right).$$

3. (10 pts) The waiting time in a certain clinic is never greater than 2 hours, and for $0 \le t \le 2$, the density function is $1 - t/2$. What is the probability of waiting less than 1 hour? What is the mean waiting time?

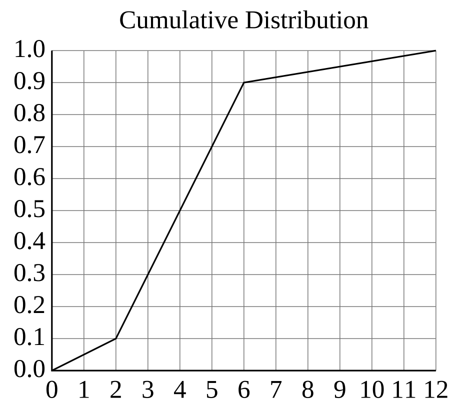
The probability of waiting 0–1 hours is

$$\int_0^1 1 - t/2\, dt = 1 - \frac{1}{4} = \frac{3}{4}.$$

The mean is given in hours by

$$\int_0^2 t(1-t/2)\,dt = \int_0^2 t - \frac{t^2}{2}\,dt = 2 - \frac{2^3}{6} = \frac{2}{3.}$$

4. (20 pts) Consider a distribution with the following cumulative distribution function:



a) Estimate the value of the density function at $x = 3$ and at $x = -1$
b) Estimate the median.
c) Do you think the mean will be greater than or less than the median?

Comparing the value at $x = 3$ to the value at $x = 4$ or $x = 2$, we see that the slope of the curve at $x = 3$ is $\approx .2$. As the cumulative distribution function is 0 and $x = 0$, it is constant at 0 for all $x < 0$, so the density function is 0. Looking for the value of x for which the cumulative distribution function is .5, we see that the median is ≈ 4. Since there a values of x much bigger than the median (up to 12) but not values much smaller than the median, the mean is larger than the median.

5. (10 pts) Evaluate the indefinite integral $\int x^2 \cos(x^3)\,dx$.
Substitute $u = x^3$, so $3x^2\,dx = du$, and the integral can be rewritten

$$\int x^2 \cos(x^3)\,dx = \int \frac{1}{3}\cos u\,du = \frac{1}{3}\sin u + C = \frac{\sin x^3}{3} + C.$$

6. (10 pts) Evaluate the indefinite integral $\int (t+1)e^{-t}\,dt$.
We use the integration by parts formula with $g'(t) = e^{-t}$ and $h(t) = t+1$. Thus $h'(t) = 1$, and (by change of variables or the formula mentioned in Problem 2) we can choose $g(t) = -e^{-t}$. Thus,

$$\int (t+1)e^{-t}\,dt = -e^{-t}(t+1) - \int -e^{-t}\,dt = -e^{-t}(t+1) - e^{-t} = -e^{-t}(t+2).$$

7. (20 pts) Evaluate the following definite integrals:

a) (10 pts) $\int_0^\pi \sin x + 2\cos x \, dx$

b) (10 pts) $\int_0^1 \frac{2x}{x^2+1} \, dx$

c) (bonus question: 5 pts) $\int_1^\infty \frac{1}{x^3} \, dx$

First we do the indefinite integrals, then plug in the endpoints and subtract.

a) $\int \sin x + 2\cos x \, dx = \int \sin x \, dx + 2\int \cos x \, dx = -\cos x + 2\sin x + C$, *so the definite integral is* $-\cos\pi + 2\sin\pi + \cos 0 - 2\sin 0 = 1 + 0 + 1 - 0 = 2$.

b) $\int \frac{2x}{x^2+1} \, dx = \ln|x^2+1| + C = \ln(x^2+1) + C$ *(by substituting* $u = x^2+1$*), so the definite integral is* $\ln 2 - \ln 1 = \ln 2$.

c) $\int_1^\infty \frac{1}{x^3} \, dx = -\frac{1}{2x^2} + C$, *so the improper integral is* $\int_1^\infty \frac{1}{x^3} \, dx = -\frac{1}{\infty} + \frac{1}{2} = \frac{1}{2}$, *where* $\frac{-1}{\infty}$ *is a not-quite-honest way of writing* $\lim_{x\to\infty} \frac{-1}{2x^2} = 0$.